2018/19 Imperial Mathematics Competition

Individual Competition
Round One

Saturday, 24 November 2018
Time allowed: 3 hours

Please fill in the lines below, writing your name in BLOCK CAPITALS.

Name: ____________________

Candidate ID: ____________________

Institution: ____________________

Signature: ____________________

Instructions:

- DO NOT open the paper until told to do so by the invigilator.
- This paper consists of 6 questions. Each question carries 10 marks.
- You may use black or blue pens only. Write clearly.
- Use separate answer sheets for each question. On each answer sheet, write down the question number, your initials and candidate ID. Use both sides wherever possible.
- You must prove any assertions you make that are not reasonably expected to be taught in the first year of an undergraduate degree programme in mathematics.
- Cross out any rough work that you would not like to be marked.
- The invigilators will collect your answer sheets as well as this cover sheet and the question paper.

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1. This questions comprises two independent parts.

   (i) Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and such that $g(0) = 0$ and $g(x)g(-x) > 0$ for any $x > 0$. Find all solutions $f : \mathbb{R} \to \mathbb{R}$ to the functional equation

   $$g(f(x+y)) = g(f(x)) + g(f(y)), \quad x, y \in \mathbb{R}.$$

   (ii) Find all continuously differentiable functions $\varphi : [a, \infty) \to \mathbb{R}$, where $a > 0$, that satisfies the equation

   $$(\varphi(x))^2 = \int_a^x (|\varphi(y)|^2 + |\varphi'(y)|^2)\,\mathrm{d}y - (x-a)^3, \quad \forall x \geqslant a.$$

2. This question, again, comprises two independent parts.

   (i) Show that if $(k+1)$ integers are chosen from $\{1, 2, 3, \ldots, 2k+1\}$, then among the chosen integers there are always two that are coprime.

   (ii) Let $A = \{1, 2, \ldots, n\}$. Prove that if $n > 11$ then there is a bijective map $f : A \to A$ with the property that, for every $a \in A$, exactly one of $f(f(f(f(a)))) = a$ and $f(f(f(f(f(a))))) = a$ holds.

3. A 'magic square' of size n is an $n \times n$ array of real numbers such that all the rows, all the columns and the two main diagonals have the same sum. Determine the dimension, over $\mathbb{R}$, of the vector space of $n \times n$ magic squares.

4. For $u, v \in \mathbb{R}^4$, let $\langle u, v \rangle$ denote the usual dot product. Define a *vector field* to be a map $\omega : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\langle \omega(z), z \rangle = 0 \ \forall z \in \mathbb{R}^4$.

   Find a maximal collection of vector fields $\{\omega_1, \ldots, \omega_k\}$ such that the map Ω sending z to $\lambda_1\omega_1(z) + \cdots + \lambda_k\omega_k(z)$, with $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, is nonzero on $\mathbb{R}^4 \backslash \{0\}$ unless $\lambda_1 = \cdots = \lambda_k = 0$.

5. For continuously differentiable function $f : [0, 1] \to \mathbb{R}$ with $f(1/2) = 0$, show that

   $$\left(\int_0^1 f(x)\,\mathrm{d}x \right)^2 \leqslant \frac{1}{4} \int_0^1 (f'(x))^2\,\mathrm{d}x$$

THE PAPER CONTINUES ON THE NEXT PAGE

6. A country has four political parties - the Blue Party, the Red Party, the Yellow Party and the Orange Party - and a parliament of 650 seats.

   (a) How many ways are there to divide the seats among the four parties so that none of the parties have a majority? (To have a majority that party must hold more than half of the seats.)

   The parliament is particularly worried about cyber security. They have decided that all login passwords must be of length exactly 6 and be a combination of a legal set of elements made up of the digits 0-9, the 52 upper and lower case letters (a-z and A-Z), and five special characters: \$, £, *, &, %. For the password to be allowed, it must contain at least one letter or special character and any letter or special character in the password must be followed by a digit (so it must end in a digit).

   (b) The Blue members of parliament have decided to choose their password by selecting 6 elements from the legal set without replacement. What is the probability it is allowed?

   Note: you may leave your answers as combinatorial or factorial terms.

THE END