IMPERIAL COLLEGE
MATHEMATICS
COMPETITION

ROUND ONE

Sunday, 24 November 2019

Name: ______________________________

Contestant Number: ______________________________

University: ______________________________

Instructions:

- Do not turn over until told to do so.
- You will have 3 hours to solve 6 problems, each of which carries 10 marks.
- Use a black or blue pen or a dark pencil. Calculators are prohibited. Rulers and compasses may be used but will not be required. Erasers are also permitted.
- Use separate answer sheets for each question. On each sheet, write down the question number, your initials and contestant number. Use both sides whenever possible.
- Problems are listed roughly in order of difficulty. Proofs are expected for all problems even if they only ask for an answer.
- One complete solution will be awarded more marks than several partial solutions.
- Do not take away the problems sheet or any rough work when leaving the venue.

Declaration:

I hereby declare that the work I am submitting is entirely my own, and that I will not discuss or disclose the contents of this paper until 24 November 2019, 18:00 GMT (25 November 2019, 05:00 AEDT).

Signature: ______________________________

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Problem	1	2	3	4	5	6	Total
Score							

Problem 1. Alice and Bob play a game on a sphere which is initially marked with a finite number of points. Alice and Bob then take turns making moves, with Alice going first:

- On Alice's move, she counts the number of marked points on the sphere, n. She then marks another $n+1$ points on the sphere.
- On Bob's move, he chooses one hemisphere and removes all marked points on that hemisphere, including any marked points on the boundary of the hemisphere.

Can Bob always guarantee that after a finite number of moves, the sphere contains no marked points?

(A *hemisphere* is the region on a sphere that lies completely on one side of any plane passing through the centre of the sphere.)

Problem 2. Find integers a and b such that

$$a^b = 3^0\binom{2020}{0} - 3^1\binom{2020}{2} + 3^2\binom{2020}{4} - \cdots + 3^{1010}\binom{2020}{2020}.$$

Problem 3. Consider a grid of points where each point is coloured either white or black, such that no two rows have the same sequence of colours and no two columns have the same sequence of colours. Let a *table* denote four points on the grid that form the vertices of a rectangle with sides parallel to those of the grid. A table is called *balanced* if one diagonal pair of points are coloured white and the other diagonal pair black.

Determine all possible values of $k \geq 2$ for which there exists a colouring of a $k \times 2019$ grid with no balanced tables.

Problem 4. Let n be a non-negative integer. Define the *decimal digit product* $D(n)$ inductively as follows:

- If n has a single decimal digit, then let $D(n) = n$.
- Otherwise let $D(n) = D(m)$, where m is the product of the decimal digits of n.

Let $P_k(1)$ be the probability that $D(i) = 1$ where i is chosen uniformly randomly from the set of integers between 1 and k (inclusive) whose decimal digit products are not 0. Compute $\lim_{k\to\infty} P_k(1)$.

Problem 5. A particle moves from the point P to the point Q in the Cartesian plane. When it passes through any point (x, y), the particle has an instantaneous speed of $\sqrt{x^2+y^2}$. Compute the minimum time required for the particle to move:

(i) from $P_1 = (-1, 0)$ to $Q_1 = (1, 0)$, and

(ii) from $P_2 = (0, 1)$ to $Q_2 = (1, 1)$.

Problem 6. Let $\varepsilon < \frac{1}{2}$ be a positive real number and let U_ε denote the set of real numbers that differ from their nearest integer by at most ε. Prove that there exists a positive integer m such that for any real number x, the sets $\{x, 2x, 3x, \ldots, mx\}$ and U_ε have at least one element in common.