IMPERIAL COLLEGE
MATHEMATICS COMPETITION

2019–2020

ROUND ONE

Official Solutions*

*We have included more than one solution where they are judged to be sufficiently different. A solution may receive full marks even if it does not appear in this booklet.

Problem 1.

(proposed by the ICMC Problem Committee)

Alice and Bob play a game on a sphere which is initially marked with a finite number of points. Alice and Bob then take turns making moves, with Alice going first:

- On Alice's move, she counts the number of marked points on the sphere, n. She then marks another $n+1$ points on the sphere.
- On Bob's move, he chooses one hemisphere and removes all marked points on that hemisphere, including any marked points on the boundary of the hemisphere.

Can Bob always guarantee that after a finite number of moves, the sphere contains no marked points?

(A *hemisphere* is the region on a sphere that lies completely on one side of any plane passing through the centre of the sphere.)

Solution.

(solution by the ICMC Problem Committee)

We claim that Bob can guarantee that the sphere contains no marked points after a finite number of moves.

Suppose that Bob draws a great circle through two marked points (this is possible in general for any two marked points). Let a denote the number of marked points on one side of the great circle, b the number of marked points on the other side, and c the number of marked points on the great circle itself. Then $a+b+c = 2n+1$, with $c \geq 2$. Adding these two inequalities yields $a+b+2c \geq 2n+3$, so by the pigeonhole principle, either $a+c$ or $b+c$ must have at least $n+2$ marked points. As these correspond to the number of marked points Bob can remove by choosing either the a or b side of great circle as the hemisphere, Bob can remove at least $n+2$ points on his turn.

If there were n points at the start of Alice's turn, there will be $2n+1$ points after Alice's move, and Bob can then guarantee that there will be at most $n-1$ points after his move. Thus, we see that after at most $2n$ moves, Bob can guarantee that there will be no marked points left on the sphere.

Problem 2. *(proposed by the ICMC Problem Committee)*

Find integers a and b such that

$$a^b = 3^0\binom{2020}{0} - 3^1\binom{2020}{2} + 3^2\binom{2020}{4} - \cdots + 3^{1010}\binom{2020}{2020}.$$

Solution. *(solution by the ICMC Problem Committee)*

Rewriting the right hand side (RHS) of the given equation as a sum of binomial expansions, we can express it as

$$\text{RHS} = \frac{(\sqrt{3}+i)^{2020} + (\sqrt{3}-i)^{2020}}{2} = \frac{(2e^{i\pi/6})^{2020} + (2e^{-i\pi/6})^{2020}}{2} = (-2)^{2019}.$$

Thus, the accepted answers are

- $a = -2,\ b = 2019,$
- $a = -8,\ b = 673,$
- $a = -2^{673},\ b = 3,$
- $a = -2^{2019},\ b = 1.$

Problem 3. *(proposed by the ICMC Problem Committee)*

Consider a grid of points where each point is coloured either white or black, such that no two rows have the same sequence of colours and no two columns have the same sequence of colours. Let a *table* denote four points on the grid that form the vertices of a rectangle with sides parallel to those of the grid. A table is called *balanced* if one diagonal pair of points are coloured white and the other diagonal pair black.

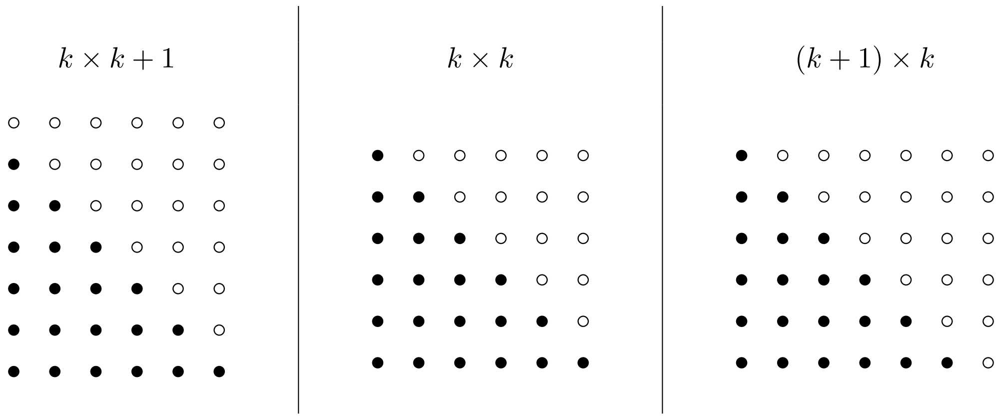
Determine all possible values of $k \geq 2$ for which there exists a colouring of a $k \times 2019$ grid with no balanced tables.

Solution. *(solution by the ICMC Problem Committee)*

We claim that the only possible values of k are 2018, 2019, and 2020. To prove this we first prove that each row and column must have a unique number of black points.

Suppose that two rows A and B each have exactly b black points, and let A_i and B_i denote the colour of the i-th point in these rows respectively. As the rows differ in at least one point, we can assume without loss of generality that A_n is white and B_n is black. Of the remaining points in A and B, the number of black points is b and $b-1$ respectively, and hence there must be an m such that A_m is black and B_m is white. Hence A_n, B_n, A_m, and B_m form a balanced table.

Thus, to avoid balanced tables, each row must have a unique number of black points, and by symmetry each column must also have a unique number of black points. Now suppose that $k \geq 2021$ (we assume that this means there are k columns and 2019 rows). Then, as each row has a number of black points that is one of $\{0, 1, 2, \ldots, 2019\}$, we see that there must be at least two rows with the same number of black points, and hence there must be a balanced table in the grid. A similar argument shows that $k \leq 2017$ is also impossible. In the cases where $k \in \{2018, 2019, 2020\}$, we see that these explicit constructions, when $k = 2019$, contain no balanced tables:



Here, any table with the top-left and bottom-right points white will also have a white top-right point, any table with the top-left and bottom-right points black will also have a black bottom-left point, so these constructions contain no balanced tables.

Problem 4. *(proposed by the ICMC Problem Committee)*

Let n be a non-negative integer. Define the *decimal digit product* $D(n)$ inductively as follows:

- If n has a single decimal digit, then let $D(n) = n$.
- Otherwise let $D(n) = D(m)$, where m is the product of the decimal digits of n.

Let $P_k(1)$ be the probability that $D(i) = 1$ where i is chosen uniformly randomly from the set of integers between 1 and k (inclusive) whose decimal digit products are not 0. Compute $\lim_{k\to\infty} P_k(1)$.

Solution. *(solution by the ICMC Problem Committee)*

We will show that $\lim_{k\to\infty} P_k(1) = 0$.

Let S be the set of non-negative integers whose decimal representation consists solely of 1's. We claim that the product of decimal digits of n is in S if and only if n itself is in S. The "if" direction is clear, so we focus on proving the "only if" direction.

We assume to the contrary that there is some non-negative integer n not in S such that the product of its decimal digits, which we call m, is in S. We note that, as n does not contain a 0, m must be uniquely expressible as $m = 2^a3^b5^c7^d$, for some non-negative integers a, b, c, and d.

If $a \geq 1$, then m would be even, and hence would not be in S, a contradiction. Similarly, if $c \geq 1$, then m would be a multiple of 5, so its last digit would be 0 or 5, another contradiction. These prove that $a = c = 0$.

Now consider the sequence $1, 11, 111, \ldots$ and so on. Modulo 3, this sequence is $1, 2, 0, 1, 2, 0, \ldots$, which is periodic with period 3. Similarly, modulo 7, this sequence is $1, 4, 6, 5, 2, 0, 1, 4, 6, 5, 2, 0, \ldots$, which is periodic with period 6. Hence if m is divisible by 3 or 7, then m must contain a number of 1's which is a multiple of 3. These integers are all divisible by 111, and in particular they are divisible by 37. Hence if $3^b7^d = m$, then $37 \mid m$, a contradiction. Hence $b = d = 0$ as well, but then $2^a3^b5^c7^d = 1$, another contradiction, as n must contain a digit which is not 1. This proves our claim.[†]

Thus, we have shown that taking the digit product of a non-negative integer n preserves the state of being an element of S. Therefore elements of S are exactly the non-negative integers with decimal digit products equal to 1. From this, we can calculate that the number of elements in S less than 10^ℓ is at most $\ell + 1$. On the other hand, the number of non-negative integers less than 10^n whose decimal digit product is non-zero is at least as large as this subset:

$$\begin{array}{ccccc}
 & & 2, & & \\
 & 12, & & 21, & \\
 & & \vdots & & \\
\underbrace{111\ldots112}_{\ell}, & \underbrace{111\ldots121}_{\ell}, & \underbrace{111\ldots211}_{\ell}, & \ldots, & \underbrace{211\ldots111}_{\ell}
\end{array}$$

This has size exactly equal to $1 + 2 + 3 + \cdots + \ell = \frac{\ell(\ell+1)}{2}$. We now note that

$$0 \leq \lim_{k\to\infty} P_k(1) = \lim_{\ell\to\infty} P_{10^\ell}(1) \leq \lim_{\ell\to\infty} \frac{\ell+1}{\frac{\ell(\ell+1)}{2}} = \lim_{\ell\to\infty} \tfrac{2}{\ell} = 0,$$

and hence by sandwich theorem, $\lim_{k\to\infty} P_k(1) = 0$ as required.

[†] In fact this claim can be proven with the help of various known results, including, but not limited to: Lifting the Exponent lemma, Fermat's little theorem, Zsigmondy's theorem, or even Catalan's conjecture.

Problem 5.

(proposed by the ICMC Problem Committee)

A particle moves from the point P to the point Q in the Cartesian plane. When it passes through any point (x, y), the particle has an instantaneous speed of $\sqrt{x^2 + y^2}$. Compute the minimum time required for the particle to move:

(i) from $P_1 = (-1, 0)$ to $Q_1 = (1, 0)$, and

(ii) from $P_2 = (0, 1)$ to $Q_2 = (1, 1)$.

Solution.

(solution by the ICMC Problem Committee)

The statement of the problem is interpreted simply as

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = x^2 + y^2.$$

Our objective is to find the "paths of shortest time" with respect to this mode of motion. Note that the origin (0, 0) can not be reached from a point other than itself in finite time, so it will be ignored from now on. In polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$), identifying θ with $\theta + 2\pi$, the equation above becomes

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \left(r\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = r^2.$$

Dividing both sides by r^2,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\log r\right)^2 + \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = 1.$$

Notice that this resembles the equation describing motion in the Cartesian plane with a constant speed of 1, with the coordinates replaced by $\log r$ and θ. Therefore, in the $(\log r)$-θ coordinate system, the paths of shortest time are the straight lines, i.e. the paths parametrised by t as

$$\begin{aligned} \log r &= at + \log r_0, \\ \theta &= bt + \theta_0, \end{aligned}$$

where $a, b, \log r_0, \theta_0$ are constants, and the speed, $\sqrt{a^2 + b^2}$, is constant and equal to 1. Thus, the shortest time required for a particle to travel between two points in the x-y coordinate system we started with is the same as the shortest time required in the $(\log r)$-θ coordinates, which in turn is equal to the length of the straight path connecting the image of the two points in the $\log r$-θ coordinate system.

(i) The points $P = (-1, 0)$ and $Q = (1, 0)$ have image

$$P = (\log r = 0,\ \theta = \pi), \qquad Q = (\log r = 0,\ \theta = 0),$$

so the minimum time required is π.

(ii) The points $P = (0, 1)$ and $Q = (1, 1)$ have image

$$P = (\log r = 0,\ \theta = \tfrac{\pi}{2}), \qquad Q = (\log r = \tfrac{1}{2}\log 2,\ \theta = \tfrac{\pi}{4}),$$

so the minimum time required is $\sqrt{\left(\frac{1}{2}\log 2\right)^2 + \left(\frac{\pi}{4}\right)^2}$.

Note: we can also derive that the paths of shortest time in the x-y coordinate system are logarithmic spirals.

Problem 6.

(proposed by the ICMC Problem Committee)

Let $\varepsilon < \frac{1}{2}$ be a positive real number and let U_ε denote the set of real numbers that differ from their nearest integer by at most ε. Prove that there exists a positive integer m such that for any real number x, the sets $\{x, 2x, 3x, \ldots, mx\}$ and U_ε have at least one element in common.

Solution 1.

(solution by the ICMC Problem Committee)

Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Fix $\varepsilon > 0$ and note that there exists $m \in \mathbb{N}$ with $\frac{1}{m-1} < \varepsilon$. Now consider the set:

$$S = \{\{x\}, \{2x\}, \ldots, \{mx\}\}$$

and the intervals

$$\left[0, \tfrac{1}{m-1}\right], \ldots, \left[\tfrac{m-2}{m-1}, 1\right].$$

Since S has m elements and there are $m-1$ intervals, by pigeonhole principle, there exist $1 \leq i < j \leq m$ such that $\{ix\}$ and $\{jx\}$ lie in the same interval $\left[\frac{k}{m-1}, \frac{k+1}{m-1}\right]$ for some $0 \leq k \leq m-2$, and hence $\{jx\} - \{ix\} \in U_\varepsilon$, which implies that $\{\{jx\} - \{ix\}\} \in U_\varepsilon$. But now note that

$$\begin{aligned}
\{\{jx\} - \{ix\}\} &= \{jx - ix - \lfloor jx \rfloor + \lfloor ix \rfloor\} \\
&= \{jx - ix\} \\
&= \{(j-i)x\}
\end{aligned}$$

and thus we have that $\{(j-i)x\} \in U_\varepsilon$, as desired.

Solution 2.

(solution by the ICMC Problem Committee)

We aim to prove that for any integer $n \geq 3$, there exists a suitable m for any $\varepsilon = \frac{1}{n}$. It suffices to prove this because for any $\varepsilon > 0$ there is some integer n such that $\frac{1}{n} < \varepsilon$. We may also assume, without loss of generality, that $x \in [0, 1]$, since U_ε has a common element with $\{x, 2x, 3x, \ldots, mx\}$ if and only if it has a common element with $\{x + r, 2(x + r), 3(x + r), \ldots, m(x + r)\}$ for any integer r. We now proceed by induction on n.

In the case where $n = 3$ (i.e. $\varepsilon = \frac{1}{3}$), we see that both of the intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$ are contained in $U_{1/3}$. Additionally, we note that for any element $x \in \left[\frac{1}{3}, \frac{2}{3}\right]$, we have that $2x \in \left[\frac{2}{3}, \frac{4}{3}\right] \subset U_{1/3}$. Hence the base case is complete.

For the inductive step, suppose that the statement holds for $n = k$, namely, that there exists a finite m_k such that $\{x, 2x, 3x, \ldots, m_k x\}$ shares a common element with $U_{1/k}$ for any x. Pick a specific value $x = x_0$. By the inductive hypothesis, $tx_0 \in U_{1/k}$ where $1 \leq t \leq m_k$ is an integer. If $tx_0 \in U_{1/(k+1)}$ then we are done, so suppose not. Then we know the distance between tx_0 and its nearest integer is in the interval $\left(\frac{1}{k+1}, \frac{1}{k}\right]$. Without loss of generality, suppose tx_0 is greater than its nearest integer, so $a + \frac{1}{k+1} < tx_0 \leq a + \frac{1}{k}$ for some integer a. Observe that $ka + 1 - \frac{1}{k+1} < ktx_0 \leq ka + 1$, and so, as ka is an integer, this proves that $ktx_0 \in U_{1/(k+1)}$.

This proves that if $\{x, 2x, 3x, \ldots, m_k x\}$ does not share an element with $U_{1/(k+1)}$, then $\{kx, 2kx, 3kx, \ldots, km_k x\}$ certainly does. Hence, the set $\{x, 2x, 3x, \ldots, km_k x\}$ has at least one element in common with $U_{1/(k+1)}$, completing our induction.