IMPERIAL COLLEGE
MATHEMATICS
COMPETITION

ROUND TWO

Sunday, 23 February 2020

Name: ____________________

Contestant Number: ____________________

University: ____________________

Instructions:

- Do not turn over until told to do so.
- You will have 3 hours to solve 4 problems, each of which carries 10 marks.
- Use a black or blue pen or a dark pencil. Calculators are prohibited. Rulers and compasses may be used but will not be required. Erasers are also permitted.
- Use separate answer sheets for each question. On each sheet, write down the question number, your initials and contestant number. Use both sides whenever possible.
- Problems are listed roughly in order of difficulty. Proofs are expected for all problems even if they only ask for an answer.
- One complete solution will be awarded more marks than several partial solutions.
- Do not take away the problems sheet or any rough work when leaving the venue.

Declaration:

I hereby declare that the work I am submitting is entirely my own, and that I will not discuss or disclose the contents of this paper until 23 February 2020, 13:00 GMT (24 February 2020, 00:00 AEDT).

Signature: ____________________

FOR MARKERS' USE ONLY					
Problem	1	2	3	4	Total
Score					

Problem 1. An *automorphism* of a group $(G, *)$ is a bijective function $f : G \to G$ satisfying $f(x * y) = f(x) * f(y)$ for all $x, y \in G$.

Find a group $(G, *)$ with fewer than $(201.6)^2 = 40642.56$ unique elements and exactly 2016^2 unique automorphisms.

Problem 2. Let $\mathbb{R}^2$ denote the set of points in the Euclidean plane. For points $A, P \in \mathbb{R}^2$ and a real number k, define the *dilation* of A about P by a factor of k as the point $P + k(A - P)$. Call a sequence of points $A_0, A_1, A_2, \ldots \in \mathbb{R}^2$ *unbounded* if the sequence of lengths $|A_0 - A_0|, |A_1 - A_0|, |A_2 - A_0|, \ldots$ has no upper bound.

Now consider n distinct points $P_0, P_1, \ldots, P_{n-1} \in \mathbb{R}^2$, and fix a real number r. Given a starting point $A_0 \in \mathbb{R}^2$, iteratively define A_{i+1} by dilating A_i about P_j by a factor of r, where j is the remainder of i when divided by n.

Prove that if $|r| \geq 1$, then for any starting point $A_0 \in \mathbb{R}^2$, the sequence $A_0, A_1, A_2, \ldots$ is either periodic or unbounded.

Problem 3. Let $\mathbb{R}$ denote the set of real numbers. A subset $S \subseteq \mathbb{R}$ is called *dense* if any non-empty open interval of $\mathbb{R}$ contains at least one element in S. For a function $f : \mathbb{R} \to \mathbb{R}$, let $\mathcal{O}_f(x)$ denote the set $\{x, f(x), f(f(x)), \ldots\}$.

(a) Is there a function $g : \mathbb{R} \to \mathbb{R}$, continuous everywhere in $\mathbb{R}$, such that $\mathcal{O}_g(x)$ is dense for all $x \in \mathbb{R}$?

(b) Is there a function $h : \mathbb{R} \to \mathbb{R}$, continuous at all but a single $x_0 \in \mathbb{R}$, such that $\mathcal{O}_h(x)$ is dense for all $x \in \mathbb{R}$?

Problem 4. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ be a set of $n \geq 2020$ distinct points on the Euclidean plane, no three of which are collinear. Andy the ant starts at some point S_{i_1} in $\mathcal{S}$ and wishes to visit a series of 2020 points $\{S_{i_1}, S_{i_2}, \ldots, S_{i_{2020}}\} \subseteq \mathcal{S}$ in order, such that $i_j > i_k$ whenever $j > k$. It is known that ants can only travel between two points in $\mathcal{S}$ in straight lines, and that an ant's path can never self-intersect.

Find a positive integer n such that Andy can always fulfil his wish.

(Lower n will be awarded more marks. Bounds for this problem may be used as a tie-breaker, should the need to do so arise.)