IMPERIAL COLLEGE
MATHEMATICS
COMPETITION

2019–2020

ROUND TWO

Official Solutions* (provisional)

*A solution may receive partial or full marks even if it does not appear in this booklet.

Problem 1. *(proposed by the ICMC Problem Committee)*

An *automorphism* of a group $(G, *)$ is a bijective function $f : G \to G$ satisfying $f(x * y) = f(x) * f(y)$ for all $x, y \in G$.

Find a group $(G, *)$ with fewer than $(201.6)^2 = 40642.56$ unique elements and exactly 2016^2 unique automorphisms.

Solution. *(solution by the ICMC Problem Committee)*

The main idea is to consider direct products of cyclic groups and compute their automorphism groups systematically under the prime factorisation $2016 = 2^5 \cdot 3^2 \cdot 7$.

For a group G, denote its set of automorphisms by $\operatorname{Aut} G$, which can clearly be endowed with the structure of a group under composition. For natural numbers m and n, denote the cyclic group of m elements by C_m, and denote the direct product of n copies of it by C_m^n. The following are several useful, though potentially superfluous, observations.

- An automorphism of C_m must map generators to generators, so that when $m = p$ is prime, $\operatorname{Aut} C_p$ is cyclic with $p - 1$ elements, and can be identified with C_{p-1}.
- In fact, without the condition that m is prime, $\operatorname{Aut} C_m$ can be identified with $C_{\Phi(m)}$, where $\Phi(m)$ is number of natural numbers less than and coprime to m.
- For a natural number k coprime to m, the direct product $\operatorname{Aut} C_k \times \operatorname{Aut} C_m$ can be identified with $\operatorname{Aut}(C_k \times C_m)$, by sending the generators of $\operatorname{Aut} C_k$ and $\operatorname{Aut} C_m$ to their product.
- If $m = p$ were prime, the direct product $\operatorname{Aut} C_p^n$ can be identified with $\mathrm{GL}(n, p)$, the group of $n \times n$ matrices with entries integers modulo p, which has exactly

  $$(p^n - p^0) \cdot (p^n - p^1) \cdot \dots \cdot (p^n - p^{n-2}) \cdot (p^n - p^{n-1})$$

  elements, by viewing C_p^n as an n-dimensional vector space over the field of p elements.

Using these observations, a group with exactly 2016^2 unique automorphisms can be reverse-engineered from its prime factorisation. First observe that

$$2016 = 12 \cdot 7 \cdot 4 \cdot 3 = 12 \cdot (2^3 - 2^0) \cdot (2^3 - 2^1) \cdot (2^3 - 2^2).$$

Then $\operatorname{Aut} C_{13}$ has 12 elements and $\operatorname{Aut} C_2^3$ has $(2^3 - 2^0) \cdot (2^3 - 2^1) \cdot (2^3 - 2^2)$ elements. Thus $\operatorname{Aut}(C_{13} \times C_2^3)$ has 2016 elements, while $C_{13} \times C_2^3$ has $13 \cdot 2^3$ elements. Next observe that

$$2016 = 42 \cdot 8 \cdot 6 = 42 \cdot (3^2 - 3^0) \cdot (3^2 - 3^1).$$

Then $\operatorname{Aut} C_{43}$ has 42 elements and $\operatorname{Aut} C_3^2$ has $(3^2 - 3^0) \cdot (3^2 - 3^1)$ elements. Thus $\operatorname{Aut}(C_{43} \times C_3^2)$ has 2016 elements, while $C_{43} \times C_3^2$ has $43 \cdot 3^2$ elements. Finally observe that

$$200^2 < 13 \cdot 2^3 \cdot 43 \cdot 3^2 < 201^2.$$

Thus $C_2^3 \times C_3^2 \times C_{13} \times C_{43}$ is a group with the desired properties.

With similar computations, one may find other groups with fewer elements but still having exactly 2016^2 automorphisms, such as $C_2^3 \times C_7^2 \times C_{13}$, $C_3^2 \times C_7^2 \times C_{43}$, or $C_2^3 \times C_3^2 \times C_{19} \times C_{29}$.

Problem 2.

(proposed by the ICMC Problem Committee)

Let $\mathbb{R}^2$ denote the set of points in the Euclidean plane. For points $A, P \in \mathbb{R}^2$ and a real number k, define the *dilation* of A about P by a factor of k as the point $P + k(A - P)$. Call a sequence of points $A_0, A_1, A_2, \ldots \in \mathbb{R}^2$ *unbounded* if the sequence of lengths $|A_0 - A_0|$, $|A_1 - A_0|$, $|A_2 - A_0|$, $\ldots$ has no upper bound.

Now consider n distinct points $P_0, P_1, \ldots, P_{n-1} \in \mathbb{R}^2$, and fix a real number r. Given a starting point $A_0 \in \mathbb{R}^2$, iteratively define A_{i+1} by dilating A_i about P_j by a factor of r, where j is the remainder of i when divided by n.

Prove that if $|r| \geq 1$, then for any starting point $A_0 \in \mathbb{R}^2$, the sequence $A_0, A_1, A_2, \ldots$ is either periodic or unbounded.

Solution.

(solution by the ICMC Problem Committee)

Let a *glide dilation* denote a dilation followed by a translation. We first show that the composition of a finite number of glide dilations is itself a glide dilation. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be glide dilations defined by

$$f(X) = P + p(X - P) + Q = \mathbf{0} + p(X - \mathbf{0}) + (1 - p)P + Q$$
$$g(X) = R + r(X - R) + S = \mathbf{0} + r(X - \mathbf{0}) + (1 - r)R + S$$

for fixed $P, Q, R, S \in \mathbb{R}^2$ and fixed real numbers p and r. Then note that

$$\begin{aligned}(f \circ g)(X) &= f(g(X)) \\ &= \mathbf{0} + pr(X - \mathbf{0}) + (1 - p)P + Q + p(1 - r)R + pS \\ &= \mathbf{0} + pr(X - \mathbf{0}) + T\end{aligned}$$

and so $f \circ g$ is a glide dilation with dilation factor equal to the product of the constituent dilations' factors, and with a translation vector equal to some $T \in \mathbb{R}^2$. By induction, the composition of any finite number of glide dilations is thus also a glide dilation, as desired.

Now fix $|r| \geq 1$, and denote the dilation of some point X about P_i by $d_i(X)$. As dilations are a subset of glide dilations, $D = d_{n-1} \circ (\cdots \circ (d_2 \circ (d_1 \circ d_0)) \cdots)$ is a glide dilation. Let the dilation factor of D be $d = r^n$, where $|r^n| \geq 1$, so that the dilation factor of $D^2 = D \circ D$ is $d^2 \geq 1$. Note that $D^2(A_i) = A_{i+2n}$ for $i \mid 2n$. There are now two cases to consider:

1. Suppose that $d^2 > 1$. Then we may write $D^2(X) = \mathbf{0} + d^2(X - \mathbf{0}) + A$ for some $A \in \mathbb{R}^2$ as

$$\begin{aligned}D^2(X) &= \mathbf{0} + d^2(X - \mathbf{0}) + \frac{(1 - d^2)A}{1 - d^2} \\ &= \frac{A}{1 - d^2} + d^2\left(X - \frac{A}{1 - d^2}\right)\end{aligned}$$

   Which is purely a dilation with dilation factor $d^2 > 1$. Hence it is clear that if $A_0 = \frac{A}{1-d^2}$ then $A_0 = A_{2n} = A_{4n} = \cdots$, so the sequence is periodic, and if not, then the subsequence $A_0, A_{2n}, A_{4n}, \ldots$ is unbounded.

2. Now suppose that $d^2 = 1$. Then we may write $D^2(X) = \mathbf{0} + (X - \mathbf{0}) + B = X + B$ for some $B \in \mathbb{R}^2$, which is purely a translation with translation vector B. Then it is clear that for any choice of A_0, the sequence is unbounded if $B \neq \mathbf{0}$ and periodic if $B = \mathbf{0}$.

Problem 3. *(proposed by Daniel Goodair and the ICMC Problem Committee)*

Let $\mathbb{R}$ denote the set of real numbers. A subset $S \subseteq \mathbb{R}$ is called *dense* if any non-empty open interval of $\mathbb{R}$ contains at least one element in S. For a function $f : \mathbb{R} \to \mathbb{R}$, let $\mathcal{O}_f(x)$ denote the set $\{x, f(x), f(f(x)), \dots\}$.

(a) Is there a function $g : \mathbb{R} \to \mathbb{R}$, continuous everywhere in $\mathbb{R}$, such that $\mathcal{O}_g(x)$ is dense for all $x \in \mathbb{R}$?

(b) Is there a function $h : \mathbb{R} \to \mathbb{R}$, continuous at all but a single $x_0 \in \mathbb{R}$, such that $\mathcal{O}_h(x)$ is dense for all $x \in \mathbb{R}$?

Solution. *(solution by the ICMC Problem Committee)*

Note that part (a) was worth 2 marks and part (b) was worth 8 marks.

(a) We prove that there is no such function. Note that the graph of a continuous function either intersects the graph of the function $f(x) = x$ or it does not. If it does, then let x_0 be the intersection point. We see that $O_g(x_0) = \{x_0\}$, a contradiction. However, if the graph of g does not intersect the graph of f, then by intermediate value theorem, it must lie completely above f or completely below f. In either case, $\mathcal{O}_g(x)$ is bounded from one side, and so cannot be dense in $\mathbb{R}$.

(b) We show that such a function exists. The idea of this construction is to take the continuous function $r : S^1 \to S^1$, where S^1 denotes the unit circle centred at the origin, which rotates a point by 1 radian anti-clockwise. We note that the orbit of any point is dense, by Problem 6 of ICMC 2019-2020 Round One. The stereographic projection from the circle to the line is continuous except for the point at infinity so that r composed with the projection is also continuous except for the point at infinity. We let the point on S^1 which maps to the point at infinity instead map to 0. We note that the densities of all orbits are preserved, and that there is exactly one discontinuity in an otherwise continuous function.

Formally, let $\tilde{f} \colon \mathbb{R} \to (0,1)$ be a homeomorphism. In particular, we can take $g \colon (-1,1) \to (0,1)$, $g(x) = \frac{x+1}{2}$ and $h \colon \mathbb{R} \to (-1,1)$, $h(x) = \frac{x}{1+|x|}$ and note that $\tilde{f} = g \circ h$ is a homeomorphism between $\mathbb{R}$ to $(0,1)$.

Fix $\alpha \in (0,1)$ not a rational and consider the circle rotation $R_\alpha \colon [0,1) \to [0,1)$ given by $R(x) = x + \alpha \mod 1$. Define:

$$f_c(x) = \begin{cases} \tilde{f}^{-1} \circ R_\alpha \circ \tilde{f} & \text{if } x \neq \tilde{f}^{-1}(1-\alpha) \\ c & \text{if } x = \tilde{f}^{-1}(1-\alpha) \end{cases}$$

where c is any value in $\mathbb{R} \backslash \{\tilde{f}^{-1}(-n \cdot \alpha \mod 1) \mid n \in \mathbb{Z}_{\geq 1}\}$.

We show that $\mathcal{O}_f(x)$ is dense in $\mathbb{R}$ for all real x. If $x \notin \{f^{-n}(\tilde{f}^{-1}(1-\alpha)) \mid n \in \mathbb{Z}_{\geq 1}\}$, then

$$f_c^n(x) = \tilde{f}^{-1} \circ R_\alpha^n(\tilde{f}(x))$$

hence $\mathcal{O}_f(x)$ is the homeomorphic image of $\mathcal{O}_{R_\alpha}$. As $\mathcal{O}_{R_\alpha}$ is dense, we have that $\mathcal{O}_f(x)$ is dense in $\mathbb{R}$. If If $x \in \{f^{-n}(\tilde{f}^{-1}(1-\alpha)) \mid n \in \mathbb{Z}_{\geq 1}\}$, by iterating f we eventually hit c and reduce the problem to the previous case. Thus, we have exhibited a family of continuous functions which are discontinuous at one point and have the required property.

Problem 4. *(proposed by the ICMC Problem Committee)*

Let $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ be a set of $n \geq 2020$ distinct points on the Euclidean plane, no three of which are collinear. Andy the ant starts at some point S_{i_1} in $\mathcal{S}$ and wishes to visit a series of 2020 points $\{S_{i_1}, S_{i_2}, \dots, S_{i_{2020}}\} \subseteq \mathcal{S}$ in order, such that $i_j > i_k$ whenever $j > k$. It is known that ants can only travel between two points in $\mathcal{S}$ in straight lines, and that an ant's path can never self-intersect.

Find a positive integer n such that Andy can always fulfil his wish.

(Lower n will be awarded more marks. Bounds for this problem may be used as a tie-breaker, should the need to do so arise.)

Solution. *(solution by the ICMC Problem Committee)*

We present the three methods known to the committee which yield a concrete bound. No other methods were successfully used by any contestant to obtain a bound differing from these. Note that perfect solutions achieving the bounds in (a), (b), and (c) score 3, 7, and 10 marks respectively.

(a) We prove the result when $n = 2^{2018} + 1$. Pick $i_1 = 1$ and $i_2 = 2$, and let $\mathcal{S}_1 = \mathcal{S}$. The line $S_{i_1}S_{i_2}$ divides the remaining points into two sets. Denote the set containing more points by $\mathcal{S}_2$, and note that it contains at least 2^{2017} points. Letting i_3 be the smallest index of the points in $\mathcal{S}_2$, note that $S_{i_2}S_{i_3}$ splits $\mathcal{S}_2$ into two sets, the larger of which will be denoted $\mathcal{S}_3$ and contain at least 2^{2016} points. Continuing in this way we note that when $S_{i_{2019}}$ is picked, $\mathcal{S}_{2019}$ will contain $2^0 = 1$ points, and hence we can pick that point to be $S_{i_{2020}}$. Note that we guarantee that the ant's path does not self-intersect as all the points $\mathcal{S}_i$ that remain on the i-th step lie on the same side of each previously-travelled line segment as the ant. Furthermore, the indices of the points visited increase by construction. Hence the ant can visit 2020 points when $n = 2^{2018} + 1$.

(b) We prove the result when $n = 2019^2 + 1$. We first introduce a system of Cartesian coordinates such that no two points in $\mathcal{S}$ have the same x-coordinate. Then, as in the statement of the lemma below, let $f(S_i)$ be the x-coordinate of S_i, and let $g(S_i) = i$. Hence the lemma guarantees the existence of a sequence of $2019 + 1 = 2020$ points in $\mathcal{S}$ such that the x-coordinate and the index are both strictly monotonic. As the path joining the sequence of points clearly cannot self-intersect, we are done if the index strictly increases. If the index strictly decreases, we can invert the order in which we consider the sequence. This does not change Andy's path (except for the direction of traversal), and the index now strictly increases, so we are done in this case also.

Lemma: Let $A = \{1, 2, \dots, a^2 + 1\}$, and let $f, g : A \to \mathbb{R}$ be injective. Then there exists a sequence $B = b_1, b_2, \dots, b_{a+1}$ of distinct elements of A such that the sequences $f(B)$ and $g(B)$ are both strictly monotonic. (This lemma is equivalent to Erdős–Szekeres Theorem.)

Proof: We may identify each integer $i \in A$ with a point in $\mathbb{R}^2$ with coordinates $(f(i), g(i))$. Then the problem is equivalent to proving that there always exists a subset of $a+1$ points which are strictly monotonic in both their x and y-coordinates. Let $C_1, C_2, \dots, C_{a^2+1}$ be the sequence of points, ordered from smallest to largest x-coordinate. Iterating through the sequence, we assign each point C_i a pair of positive integers: firstly, the length of the longest decreasing subsequence† of points ending with C_i, and secondly, the length of the

†Decreasing with respect to the y-coordinate only.

longest increasing subsequence of points ending with C_i. We call this pair of numbers the *signature* of the point. Now note that no two points can have the same signature, as the point with greater x-coordinate must be either higher or lower than the point with lesser x-coordinate. However, there are only a^2 possible signatures if neither component in the signature of any point exceeds a, a contradiction. Hence at least one point must have a signature with one component greater than a, and the result follows.

(c) We prove the result when $n = 2018^2 + 2$.‡ Let S_k be a point lying on the convex hull of $\mathcal{S}$. Let $\mathcal{S}' = \mathcal{S} \setminus \{S_k\}$, and let X be the point adjacent to S_k on the convex hull of $\mathcal{S}$ going anti-clockwise. Now introduce a system of polar coordinates with the origin at S_k such that X lies on the ray $\theta = 0$. Then, as in the statement of the lemma above, let $f(S_i)$ equal the argument of S_i and let $g(S_i) = i$, for all $S_i \in \mathcal{S}'$. Hence the lemma guarantees the existence of a sequence of $2018 + 1 = 2019$ points in $\mathcal{S}'$ such that the argument and the index are both strictly monotonic. Note that the argument is a strictly monotone function of distance travelled along the path, but as it is contained in $[0, \pi)$, the function must be injective, and hence the path cannot self-intersect. We now apply the same sequence-inverting argument as in part (b) to correct the direction of traversal if it is flipped. Denote the sequence of 2019 points in its current state $S_{i_1}, \ldots, S_{i_{2019}}$.

Finally, note that due to the path strictly increasing argument, each of the triangles $S_k S_{i_j} S_{i_{j+1}}$ for $j \in \{1, \ldots, 2018\}$ contains no part of the path in its interior. Now fixing j such that $i_j < k < i_{j+1}$, where we let $i_0 = 0$ and $i_{2019} = 2018^2 + 3$, we may simply modify the existing path by inserting S_k as a detour between S_{i_j} and $S_{i_{j+1}}$ to obtain the required sequence of 2020 points.

‡This is the best bound known to the ICMC Problem Committee.