IMPERIAL COLLEGE
MATHEMATICS
COMPETITION

2020–2021

ROUND ONE

Official Solutions*

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*A solution may receive full or partial marks even if it does not appear in this booklet.

Problem 1. *(proposed by Tony Wang)*

A set of points in the plane is called *sane* if no three points are collinear and the angle between any three distinct points is a rational number of degrees.

(a) Does there exist a countably infinite sane set $\mathcal{P}$?

(b) Does there exist an uncountably infinite sane set $\mathcal{Q}$?

Notes on Marking. Five marks were allocated to part (a), and four to part (b). The last mark was given for stating the correct answer to *both* parts. A correct construction for part (a) without a proof was awarded three marks. Some contestants proved the result only for an arbitrarily large finite number of points – this was awarded one mark.

Solution 1. *(solution by Tony Wang)*

We prove that the answers to the two parts are yes and no, respectively.

(a) We use directed angles for convenience. Let Γ be a circle with center O. Let A be a point on Γ. We define $\mathcal{P}$ to be the set of all points P on the unit circle where $\measuredangle AOP$ is rational. Note that $\mathcal{P}$ has a countably infinite number of points. We now show that for any distinct $B, C, D \in \mathcal{P}$, $\measuredangle BCD$ is rational: Note that by elementary circle theorems, $\measuredangle BCD = \frac{1}{2}\measuredangle BOD = \frac{1}{2}(\measuredangle BOA + \measuredangle AOD)$, where the summands are rational, and hence $\measuredangle BCD$ is rational, as desired. $\square$

(b) Suppose for the sake of contradiction that there exists an uncountably infinite sane set $\mathcal{Q}$. Fix two points $P, Q \in \mathcal{Q}$. Then any other point $R \in \mathcal{Q}$ must lie on a line ℓ passing through Q such that the angle formed between ℓ and PQ is rational. However, note that there are only a countably infinite number of possible ℓ, and hence by infinite pigeonhole principle, one such line must contain a uncountably infinite number of points in $\mathcal{Q}$, but this contradicts the fact that no three points are collinear. Hence no such $\mathcal{Q}$ exists. $\square$

Solution 2. *(solution by the ICMC Problem Committee)*

For part (a) we use the same solution as above. For part (b):

(b) Suppose for the sake of contradiction that there exists an uncountably infinite sane set $\mathcal{Q}$. Fix two points $P, R \in \mathcal{Q}$. Then any other point $Q \in \mathcal{Q}$ must satisfy the condition that $\angle PQR$ is rational. However, there are only a countably infinite number of different possible angles for $\angle PQR$, and so by infinite pigeonhole principle, a particular rational q must be the angle for an uncountably infinite number of such $\angle PQR$'s. The locus of points defined by Q such that $\angle PQR = q$ forms two incomplete circles. There must be an uncountably infinite number of points on one of these circles.

Let the circle be Γ and its centre O. As there are an uncountbaly infinite number of points in $\Gamma \cap \mathcal{Q}$, there must exist a point $A \in \Gamma \cap \mathcal{Q}$ such that $\angle POA$ is irrational. Hence by the same argument used in part (a), we can show that $\angle PQA$ must also be irrational, a contradiction. $\square$

Problem 2. *(proposed by Tony Wang)*

Let A be a square matrix with entries in the field $\mathbb{Z}/p\mathbb{Z}$ such that $A^n - I$ is invertible for every positive integer n. Prove that there exists a positive integer m such that $A^m = 0$.

(A matrix having entries in the field $\mathbb{Z}/p\mathbb{Z}$ means that two matrices are considered the same if each pair of corresponding entries differ by a multiple of p.)

Notes on Marking. No marks were awarded for proving that A is not invertible or has determinant 0. There was no penalty for stating the incorrect finite number of matrices. In solution 2, a common mistake was to assume that all eigenvalues of A were in $\mathbb{Z}/p\mathbb{Z}$. Not all matrices in $\mathbb{Z}/p\mathbb{Z}$ have eigenvalues in $\mathbb{Z}/p\mathbb{Z}$, and working over the algebraic closure was critical. As such, there was 5 point penalty for neglecting to work in the algebraic closure. Many contestants incorrectly assumed various cancellation laws. Note that in the context of the problem, a series expansion of $(I - A^n)^{-1}$ is not well-defined unless A is nilpotent.

Solution 1. *(solution by Tony Wang)*

Note that there are finitely many matrices under consideration. Let the number be $k \in \mathbb{N}$. Then by pigeonhole principle, two of the $k+1$ matrices $A, A^2, A^3, \dots, A^{k+1}$ must be equal. Suppose that they are A^r and A^s, where $r > s$. Then note that

$$\begin{aligned} A^r - A^s = 0 &\iff A^s(A^{r-s} - I) = 0 \\ &\iff A^s = 0, \end{aligned}$$

since $A^{r-s} - I$ is invertible. □

Solution 2. *(solution by Cristi Calin)*

We shall prove the statement using eigenvalues and the Cayley-Hamilton theorem.

By the Cayley-Hamilton theorem, the characteristic polynomial c_A of A satisfies $c_A(A) = 0$. Consider the eigenvalues of A over the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$. The closure is countable, but all elements have finite order. Suppose for the sake of contradiction that some eigenvalue λ is not zero. Then there exists $m \in \mathbb{N}$ such that $\lambda^m = 1$. However this implies $\det(A^m - I) = 0$, and $A^m - I$ is therefore not invertible, which is a contradiction.

Since all eigenvalues are 0, the characteristic polynomial must be $c_A = X^n$. Thus, $c_A(A) = A^n = 0$, as required. □

Problem 3. *(proposed by Cristi Calin)*

Let $s_n = \displaystyle\int_0^1 \sin^n(nx)\,dx$.

(a) Prove that $s_n \leq \dfrac{2}{n}$ for all odd n.

(b) Find all the limit points of the sequence $s_1, s_2, s_3, \ldots$.

Notes on Marking. Four marks were awarded for solving part (a) completely. Another mark was awarded for stating the correct answer for part (b). No marks were awarded for the substitution $x \mapsto x/n$. A brief graphical explanation of the proof for part (b) as shown in Solution 1 was awarded few marks. Some unique bounding arguments were awarded full marks, as well as a few solutions which used Wallis' integrals or the dominated convergence theorem.

Solution 1. *(solution by the ICMC Problem Selection Committee)*

We proceed with each part as follows:

(a) By the substitution $x \mapsto x/n$ we may consider $s_n = \frac{1}{n}\int_0^n \sin^n(x)\,dx$. Note that because $\sin(x) = -\sin(x+\pi)$, we have $\sin^n(x) = -\sin^n(x+\pi)$ for odd n. Let $k = n \mod 2\pi$. By the parity of $\sin^n(x)$, we know that $\int_0^{2\pi} \sin^n(x)dx = 0$. Hence,

$$\frac{1}{n}\int_0^n \sin^n(x)\,dx = \frac{1}{n}\int_0^k \sin^n(x)\,dx \leq \frac{1}{n}\int_0^\pi \sin^n(x)\,dx \leq \frac{1}{n}\int_0^\pi \sin(x)\,dx = \frac{2}{n},$$

as required. □

(b) We shall prove that the only limit point is 0. We already know that $s_n \to 0$ for odd n, so it suffices to prove that $s_n \to 0$ for even n. To do so, we will bound the function from above using rectangles, i.e. we shall take an appropriate upper Darboux sum to bound $\int_0^n \sin^n(x)\,dx$.

For a sufficiently small $\delta > 0$, we define the partition

$$P_\delta(n) = \left\{\left[0, \frac{\pi}{2} - \delta\right), \left[\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta\right), \left[\frac{\pi}{2} + \delta, \frac{3\pi}{2} - \delta\right), \ldots, \left[\frac{X\pi}{2} \pm \delta, n\right]\right\}.$$

For each interval in $P_\delta(n)$ containing some multiple of $\pi/2$, we bound $\sin^n(x)$ by a rectangle of height 1. Otherwise, we can bound the interval by $\sin^n(x)$ by a rectangle of height $\sin^n(\pi/2 - \delta) = \cos^n(\delta)$, where we note that $\cos^n(\delta) \to 0$ as $n \to \infty$. We note that there are less than n multiples of $\pi/2$ in the interval $[0, n]$, so the sum of the areas of these rectangles is at most $2n\delta + n\cos^n(\delta)$, and hence

$$s_n \leq \frac{1}{n}\left(2n\delta + n\cos^n(\delta)\right) = 2\delta + \cos^n(\delta) \to 2\delta, \quad \text{as } n \to \infty.$$

Now suppose for the sake of contradiction that $2\varepsilon > 0$ was a limit point of the sequence. By selecting $\delta < \varepsilon$, we show that the terms of s_n are eventually inside the interval $(0, 2\varepsilon)$, a contradiction. Then the sequence is bounded in the interval $[0, 1]$ but has no limit point greater than 0, hence by Bolzano–Weierstrass theorem, the only limit point is 0. □

Solution 2. *(solution by Cristi Calin)*

Let $y = nx$, so that $dy = n \cdot dx$. The integral a_n can therefore be rewritten as

$$a_n = \int_0^1 \sin^n(nx) = \frac{\int_0^n \sin^n(y)dy}{n}.$$

We know from Euler's identity that $\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$, so $\sin^n(y) = \frac{(e^{iy}-e^{-iy})^n}{(2i)^n}$. The numerator can be rewritten using the Binomial formula as

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} e^{iky} \cdot e^{-i(n-k)y} = \sum_{k=0}^{n} \binom{n}{k} e^{(2k-n)iy} \cdot (-1)^{n-k}.$$

Thus,

$$\begin{aligned} a_n &= \frac{\int_0^n \sum_{k=0}^n \binom{n}{k}(-1)^{n-k} \cdot e^{(2k-n)iy} dy}{(2i)^n} \\ &= \frac{\sum_{k=0}^n \int_0^n \binom{n}{k}(-1)^{n-k} \cdot e^{(2k-n)iy} dy}{(2i)^n}. \end{aligned}$$

We now look separately at the cases when n is odd and when n is even.

- **Case 1:** When n is odd, the numerator can be rewritten as $\sum_0^n \binom{n}{k} \cdot (-1)^{n-k} \cdot \frac{e^{(2k-n)in}-1}{2k-n}$ as $2k - n \neq 0$. We apply the triangle inequality to $|a_n|$, so that we get

$$|a_n| \leq \frac{1}{2^n \cdot n} \cdot \sum_{k=0}^{n} \binom{n}{k} \cdot \left| \frac{e^{(2k-n)in} - 1}{2k-n} \right| \leq \frac{1}{2^n \cdot n} \cdot \sum_{k=0}^{n} \binom{n}{k} \cdot 2 = \frac{2}{n} \to 0.$$

- **Case 2:** When n is even, for $k = \frac{n}{2}$, $e^{(2k-n)iy} = 1$, and its integral from 0 to n is n. Thus, the numerator of a_n is equal to

$$\sum_{\substack{k=0 \\ k \neq n/2}}^{n} \binom{n}{k} \cdot (-1)^{n-k} \cdot \frac{e^{(2k-n)in} - 1}{2k-n} + \binom{n}{n/2} \cdot n.$$

  When we divide and apply the triangle inequality again, it becomes a sum of 2 terms which both tend towards 0, so $a_n \to 0$. To see that the term $\binom{n}{n/2}/2^n \to 0$, it is easy to prove by induction that it is smaller than $\frac{1}{\sqrt{n+1}}$, which solves the problem. $\square$

Problem 4. *(proposed by Tony Wang)*

Does there exist a set $\mathcal{R}$ of positive rational numbers such that every positive rational number is the sum of the elements of a unique finite subset of $\mathcal{R}$?

Notes on Marking. To clarify, the sums involved cannot contain repeated elements of $\mathcal{R}$. No marks were awarded for stating the correct answer, or for analysing the cardinality of sets.

Solution 1. *(solution by Tony Wang)*

We show that the answer is no, by assuming for the sake of contradiction that there does exist such a set $\mathcal{R}$.

The set must be non-empty. Since scaling all values in $\mathcal{R}$ has no effect on the sum of elements of subsets, we may assume WLOG that $1 \in \mathcal{R}$. Now suppose for the sake of contradiction that $x, y \in \mathcal{R}$ with $x \in (y, 2y)$. Then note that $x - y$ is the sum of a unique subset $A \subset \mathcal{R}$, but $y \notin A$ as $y > x - y$. Hence $\{x\}$ and $\{y\} \cup A$ are two distinct sets whose elements sum to x, a contradiction. Hence there cannot exist two elements of $\mathcal{R}$ which are a factor of less than 2 apart, a lemma we will use in the following paragraph.

Now suppose that $r_1, r_2, r_3, \ldots$ are the elements of $\mathcal{R}$ in the interval $(0, 1)$. By the lemma above, there only exist a finite number of elements in any interval (a, b) for $a > 0$, and so we may assume they are ordered $r_1 > r_2 > r_3 > \cdots$ without loss of generality. Note that $1/2^n \geq r_n$, otherwise we contradict the lemma above. However, if for any m, $r_m = 1/2^m - \varepsilon$ for $\varepsilon \in (0, 1/2^m)$, then the number $1 - \varepsilon/2$ would become unattainable as a sum of the elements of a finite subset of $\mathcal{R}$, since

$$\sup_{A \subset \mathcal{R} \cap (0,1)} \leq \sum_{i=1}^{\infty} r_i \leq \sum_{i=1}^{\infty} \frac{1}{2^i} - \varepsilon = 1 - \varepsilon.$$

Hence we must have $r_m = 1/2^m$ for all m. But now $1/3 = 0.\overline{01}_2$ is not expressible as the finite sum of powers of two, and hence as the finite sum of a subset of $\mathcal{R}$, and so such a set $\mathcal{R}$ cannot exist. $\square$

Solution 2. *(solution by Harun Khan)*

Similarly to the first solution, we show that the answer is no, by assuming for the sake of contradiction that there does exist such a set $\mathcal{R}$. Similarly, the set must be non-empty, so we assume WLOG that $1 \in \mathcal{R}$ and we prove that if $y \in \mathcal{R}$, then $(y, 2y) \cap \mathcal{R} = \emptyset$.

We will show that if $y \in \mathcal{R}$, then $2y \in \mathcal{R}$. Assume for a contradiction that $2y \notin \mathcal{R}$. Then $2y$ can be written as a sum of the elements of a finite subset of $\mathcal{R}$, say $A = \{x_1, x_2, ..., x_n\}$. If $y \in A$, then $A \backslash \{y\}$ and $\{y\}$ will be two finite subsets of $\mathcal{R}$ that sum to y, a contradiction. Hence $y \notin A$. Since $2y \notin \mathcal{R}$, we must have that $n \geq 2$, so there exists at least one element of A which is strictly smaller than y. So assume for a contradiction that $x_1 < y$. Then $2y - x_1 = \sum_{k=2}^{n} x_k$ is one representation of $2y - x_1$. Since $x_1 < y$, we have that $y - x_1 > 0$, so suppose that B is a subset of $\mathcal{R}$ that sums to $y - x_1$. Then $y \notin B$ (as $y - x_1 < y$), so $B \cup \{y\}$ and $A \backslash \{x_1\}$ are two representations of $2y - x_1$ (which are different since one contains y, while the other doesn't). This shows a contradiction, so $2y$ must be in $\mathcal{R}$. A simple inductive argument shows that if $y \in \mathcal{R}$, then for any $n \in \mathbb{Z}_{\geq 1}$, $2^n y \in \mathcal{R}$.

Now let $c \in \mathcal{R}$ such that $c < 1$. Then there exists $n \in \mathbb{Z}_{\geq 1}$ such that $c \in [1/2^n, 1/2^{n-1})$. The above paragraph implies that $2^n c \in \mathcal{R}$, but $2^n c \in [1, 2)$, so by the first paragraph $2^n c = 1$. Hence the only possible elements of $\mathcal{R}$ smaller than 1 are negative powers of 2. Similarly as in the first solution, $1/3$ is not expressible as a finite sum of powers of two, which shows a contradiction, so no such set $\mathcal{R}$ can exist. $\square$

Problem 5.

(proposed by Harun Khan)

Find all composite positive integers m such that, whenever the product of two positive integers a and b is m, their sum is a power of 2.

Notes on Marking. No marks were awarded for stating the correct answer, deriving that $m = 2^k - 1$ for some positive integer k, or proving that $ab + 1 \geq a + b$. However, in some cases, a mark was deducted for omitting the proof that $ab + 1 \geq a + b$. Two marks were awarded for proving that, apart from $k = 4$, k cannot be even (or composite), but in many cases there were algebraic or bounding errors, and one mark was deducted. Some contestants working through solution 4 incorrectly concluded that if $2^{k-1} \equiv 1 \pmod p$ and $2^{p-1} \equiv 1 \pmod p$, then $k - 1 \mid p - 1$. As a counterexample, consider that $2^{25} \equiv 1 \pmod{31}$, but $25 \nmid 30$.

Solution 1.

(solution by the ICMC Problem Selection Committee)

Let $b \geq a > 1$ be positive integers and let $ab + 1 = 2^n$ and $a + b = 2^k$. Since $ab = 2^n - 1$ must be odd, a and b must both be odd. Additionally, $(ab + 1) - (a + b) = (a - 1)(b - 1) \geq 0$, so we have $2^n \geq 2^k$, and hence

$$\begin{aligned} 2^k \mid 2^n &\implies a + b \mid ab + 1 \\ &\implies a + b \mid ab + 1 - a(a + b) \\ &\implies 2^k \mid (a + 1)(a - 1). \end{aligned}$$

Since a is odd, $\gcd(a + 1, a - 1) = 2$, and so either $2^{k-1} \mid a + 1$ or $2^{k-1} \mid a - 1$. However, $2^{k-1} = \frac{a+b}{2} \geq \frac{a+a}{2} > a - 1$, so we must have $2^{k-1} = a + 1$, or $a = 2^{k-1} - 1$. Since $a + b = 2^k$, we must also have $b = 2^{k-1} + 1$. In particular, note that a and b differ by two.

Finally, since $ab = 4^{k-1} - 1 \equiv 0 \pmod 3$, we have $m = 3 \times \frac{ab}{3}$. However, as $\frac{ab}{3}$ differs from 3 by 2, the only solution is $m = 15 = 3 \times 5$, which is easily confirmed. □

Solution 2.

(solution by Tony Wang)

Clearly, $m = 2^p - 1$ for some integer p. We can check that $p = 1, 2, 3$ don't work, and that $p = 4$ does work. If $p > 4$ is not prime, then let $p = qr$ where $q \geq r \geq 2$. Then $q \geq 3$, and in base 2,

$$2^p - 1 = \underbrace{1111\cdots1}_{p}{}_2 = \underbrace{\underbrace{00\cdots01}_{r}\underbrace{00\cdots01}_{r}\cdots\underbrace{00\cdots01}_{r}}_{q}{}_2 \times \underbrace{111\cdots1}_{r}{}_2.$$

Then as $q \geq 3$, adding up the factors yields a binary number with at least two 1's, which cannot be a power of 2. The only remaining cases are where $p > 4$ is prime.

Let q be a prime which divides $2^p - 1$. Then $2^p \equiv 1 \pmod q$, and by Fermat's little theorem, $2^{q-1} \equiv 1 \pmod q$. Hence $2^{\gcd(p,q-1)} \equiv 1 \pmod q$, a contradiction unless $p \mid q - 1$ which implies $q = cp + 1$. As this is true for any prime factor q of $2^p - 1$, we can write

$$2^p - 1 = (cp + 1)(dp + 1) = cdp^2 + (c + d)p + 1, \quad \text{for positive integers } c \text{ and } d.$$

Writing $2^k = (c + d)p + 2$, we have $2^k \mid 2^p = cdp^2 + (c + d)p + 2 \implies (c + d)p + 2 \mid cd$.

Now note that if $\nu_2(c + d) \neq 1$, then $k = \nu_2((c + d)p + 2) = 0$ or 1, a contradiction, and so $\nu_2(c + d) = 1$. If c and d are odd, then cd is odd, a contradiction. So, WLOG, to fulfil $\nu_2(c+d) = 1$ we must have $\nu_2(c) = 1$ and $\nu_2(d) = e > 1$. Then $2^k = (c+d)p+2 \geq (2+2^e)p+2$, and so $k > e+1$, but since $2^k \mid cd$, and we have $\nu_2(cd) = e+1$, we obtain our final contradiction.

Hence the only solution is $m = 15$. □

Problem 6. *(proposed by Tony Wang)*

There are $n + 1$ squares in a row, labelled from 0 to n. Tony starts with k stones on square 0. On each move, he may choose a stone and advance the stone up to m squares where m is the number of stones on the same square (including itself) or behind it.

Tony's goal is to get all stones to square n. Show that Tony cannot achieve his goal in fewer than $\frac{n}{1} + \frac{n}{2} + \cdots + \frac{n}{k}$ moves.

Notes on Marking. Many contestants attempted to use a greedy algorithm or induction. In many cases these were awarded zero marks. Consider that when there are 7 squares and 3 stones, it is possible for Tony to achieve his goal in 13 moves, which provides a counterexample to any proofs which allow the bound to be improved to $\lceil \frac{n}{1} \rceil + \lceil \frac{n}{2} \rceil + \cdots + \lceil \frac{n}{k} \rceil$ moves. There also exist counterexamples to many proofs which attempted to bound the number of jumps of certain lengths.

Solution 1. *(solution by Tony Wang)*

For each move, the *score* of the move is calculated as follows: we shift the stone to the right one square at a time. For each shift, the score of that shift by $1/i$, where i is the number of stones on the same square (including itself) or behind it. The score of the move is the sum of the scores of the shifts of the move. We attribute the score of a shift as being *given* by the square the stone started on at the beginning of the shift.

Note that the score of each move is at most 1: If we pick a stone to move, that can move up to i squares, then the score of the first shift is $1/i$. The score of each shift does not increase thereafter, and so the sum of the scores of the shifts is at most 1.

Note that the total score of all moves will be $n/1 + n/2 + \cdots + n/k$: For each of the first n squares, all k stones must eventually pass it, so the sum of the scores given by each square must be equal to $1/1 + 1/2 + \cdots + 1/k$. This is multiplied by the number of squares passed to finish the proof. $\square$

Solution 2. *(solution by Harun Khan)*

After every move, we label every stone so that the i^{th} stone from the left is labelled i. After move m, suppose that the stone i is on square s_i for $i \in \{1, 2, \ldots, k\}$. Let the *score*

$$S_m = \frac{s_1}{1} + \frac{s_2}{2} + \cdots + \frac{s_k}{k}.$$

We show that $S_{m+1} - S_m \leq 1$. Indeed, some stone r is moved $d \leq r$ squares in the $(m+1)$-th move. If this stone hasn't been relabelled, then clearly, $S_{m+1} - S_m \leq 1$ since it moves at most r moves yet its weight remains $1/r$. On the other hand, if stone r is re-labelled to $r + l$, then as the stones were ordered:

$$\begin{aligned}
S_{m+1} - S_m &= \frac{a_r + d}{r + l} + \sum_{j=r}^{r+l-1} \frac{a_{j+1}}{j} - \sum_{j=r}^{r+l} \frac{a_j}{j} \\
&= \frac{a_r + d - a_{r+l}}{r + l} + \sum_{j=r}^{r+l-1} \frac{a_{j+1} - a_j}{j} \\
&< \frac{a_r + d - a_{r+l}}{r} + \sum_{j=r}^{r+l-1} \frac{a_{j+1} - a_j}{r} \\
&= \frac{d}{r} \leq 1.
\end{aligned}$$

Now note that $S_0 = 0$ and $S_z = \frac{n}{1} + \frac{n}{2} + \cdots + \frac{n}{k}$, where Tony takes z moves to achieve his goal. As the score can increase by at most 1 on each move, he requires at least S_z moves. □

Remark. One can use the rearrangement inequality instead to prove that our monovariant increases by at most 1 after every move.

Solution 3.

(solution by Tony Wang)

Construct a $k \times (n + 1)$ grid (matrix notation). Each column represents a square. In the cell a_{ij}, write the number $1/i$. At the start, each stone is on a cell a_{i0} (the stones fill up the leftmost column). If we pick a stone on a square i and move it to the right j squares, this corresponds to picking any stone on column i in the grid and shifting it to the right j cells, without changing the row of the stone.

On any move, we ensure the following two conditions are satisfied:

- On each row there is exactly one stone.
- The stone on row i is further right or on the same column as the stone on row $i - 1$.

The first condition is trivially satisfied by the way we shift the stones in the grid. The second condition is satisfied thus: The base case is trivial. On each move thereafter, we shift a chosen stone in our grid to the right, one cell at a time, until the next shift would violate the condition. At this point, there would be a stack of at least one stone directly underneath the stone we are shifting on the grid. We switch to shifting the bottom-most stone of that stack instead. This process repeats until the move is complete. We note that the final position on our grid (at the end of the move) still corresponds to the position of the stones on the squares (at the end of the move), even though our process of shifting the stones on our grid may be convoluted.

Because of the two conditions, we may say that on any move, the stone on row i can shift at most i squares on our grid. In particular, if we start the move by shifting the stone on row i, the total rightwards movement of all stones in our grid is at most i for that move, and furthermore, no stones above row i will be shifted. The numbers written on these squares are all equal to $1/i$ or less, and so the total "score" for any move is at most 1. But note that the total score increase after all moves have been played is $n/1 + n/2 + \cdots + n/k$, and so that is the minimum number of moves required. □