IMPERIAL COLLEGE
MATHEMATICS
COMPETITION

ROUND TWO

Sunday, 28 February 2021

Instructions:

- As far as possible, the exam should be taken under test conditions, in a quiet room, with only the permitted materials and a printout of the exam if desired.
- You will have 3 hours to solve 4 problems, each of which carries 10 marks. You will have 30 minutes after the competition to upload your solutions.
- You are recommended to use a black or blue pen or a dark pencil. Rulers, compasses, and erasers may be used but will not be required. All electronic devices, including calculators, are prohibited, except to view this paper, email our committee, or type up your solutions in a word processor or TeX editor.
- Write your Contestant ID on every page. We will require a separate file upload for each problem. Do not include any other personally identifiable information such as your name in your scripts or filenames.
- Problems are listed roughly in order of difficulty. Proofs are expected for all problems even if they only ask for an answer.

If you encounter difficulties or wish to ask a question about the paper or the problems, please contact us at `contest@icmathscomp.org`. We will try to get back to you as quickly as possible.

Problem 1. Let S be a set with 10 distinct elements. A set T of subsets of S (possibly containing the empty set) is called *union-closed* if, for all $A, B \in T$, it is true that $A \cup B \in T$. Show that the number of union-closed sets T is less than 2^{1023}.

(A bound of 2^{1023} will be awarded full marks, but lower bounds for this problem may be used as a tie-breaker for the competition.)

Problem 2. Let $p > 3$ be a prime number. A sequence of $p-1$ integers $a_1, a_2, \ldots, a_{p-1}$ is called *wonky* if they are distinct modulo p and $a_i a_{i+2} \not\equiv a_{i+1}^2 \pmod{p}$ for all $i \in \{1, 2, \ldots, p-1\}$, where $a_p = a_1$ and $a_{p+1} = a_2$. Does there always exist a wonky sequence such that

$$a_1a_2, \qquad a_1a_2 + a_2a_3, \qquad \ldots, \qquad a_1a_2 + \cdots + a_{p-1}a_1,$$

are all distinct modulo p?

Problem 3. Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be continuous functions and X be a random variable such that $E(g(X)h(X)) = 0$ and $E(g(X)^2) \neq 0 \neq E(h(X)^2)$. Prove that

$$E(f(X)^2) \geq \frac{E(f(X)g(X))^2}{E(g(X)^2)} + \frac{E(f(X)h(X))^2}{E(h(X)^2)}.$$

You may assume that all expected values exist.

Problem 4. Let $\mathbb{R}^2$ denote the Euclidean plane. A continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$ maps circles to circles. (A point is not a circle.) Prove that it maps lines to lines.