IMPERIAL COLLEGE
MATHEMATICS
COMPETITION

2020–2021

ROUND TWO

Official Solutions*

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*A solution may receive full or partial marks even if it does not appear in this booklet.

Video Solutions

J Pi Maths, a YouTube channel run by a contestant this year, has made video solutions for all the Round 1 and 2 problems this year. You can check them out here!

Problem 1. *(proposed by Tony Wang)*

Let S be a set with 10 distinct elements. A set T of subsets of S (possibly containing the empty set) is called *union-closed* if, for all $A, B \in T$, it is true that $A \cup B \in T$. Show that the number of union-closed sets T is less than 2^{1023}.

(A bound of 2^{1023} will be awarded full marks, but lower bounds for this problem may be used as a tie-breaker for the competition.)

Notes on Marking. The markers decided to condone a bound of exactly 2^{1023}.

The solutions presented here achieve the following bounds, respectively:

$$\begin{aligned} 14^{256} &\approx 2^{974.7} \\ \textstyle\sum_{r=0}^{10} \binom{10}{r} 2^{1015+r-2^r} &\approx 2^{1019.4} \\ 2^{1023} - 1 &\approx 2^{1023} \\ 2^{1023} - 2^{1018} + 2^{87} &\approx 2^{1023} \\ 122^{128} &\approx 2^{887.1} \end{aligned}$$

Solution 5 is very similar to solution 1 except that it uses the base case $\{1, 2, 3\}$ instead of $\{1, 2\}$, resulting in a larger computation but a lower bound.

Solution 1. *(solution by contestants)*

Let $f(n)$ denote the number of union-closed sets T over a set $S_n = \{1, 2, \ldots, n\}$. We bound $f(10)$ by showing that $f(n) \leq f(n-1)^2$.

Let T be a union-closed set over S_n, for $n \geq 1$. We partition T into A and B, where $A = \{t \in T : n \notin t\}$, and $B = \{t \in T : n \in t\}$. Then define $B' = \{b \backslash \{n\} : b \in B\}$. Note that both A and B' are union-closed sets over S_{n-1}, and furthermore, T uniquely partitions into A and B' by the above method. Since there are $f(n-1)^2$ choices of A and B', we have $f(n) \leq f(n-1)^2$, as desired (we have an inequality here since not all T generated by arbitrary A and B' will be union-closed over S_n).

We now count that $f(2) \leq 14$, since $|\mathcal{P}(\mathcal{P}(S_2))| = 16$, but $\{\{1\}, \{2\}\}$ and $\{\varnothing, \{1\}, \{2\}\}$ are not union-closed. Hence $f(10) \leq f(2)^{2^8} = 14^{256} < 2^{1023}$, since $(14/16)^{256} < 1/2$.

Solution 2. *(solution by contestants)*

Call a *singleton* a subset of S with exactly one element. Let T be a union-closed set with exactly r singletons. If the singletons are $\{1\}, \{2\}, \ldots, \{r\}$, then $\{r+1\}, \ldots, \{10\} \notin T$ so $10 - r$ elements are not in T.

Let $A = \{1, 2, \ldots, r\}$. On the other hand, we claim $\mathcal{P}(A) \backslash \{\varnothing\} \subset T$. Indeed $\{1\}, \{2\}, \ldots, \{r\} \in T$. Now suppose all k element subsets of A are in T. Then any $k+1$-element subset of A can be written as a union of a k-element subset and a 1-element subset, and so is an element of T. Hence every element of $\mathcal{P}(A) \backslash \{\phi\}$ is an element of T. So the number of union-closed sets with exactly r singletons is less than $2^{2^{10}-(10-r+2^r-1)}$ since $10 - r + 2^r - 1$ of 1024 choices are already determined. Hence the number of union-closed sets is at most

$$\sum_{r=0}^{10}\binom{10}{r}2^{2^{10}-(10-r+2^r-1)} < 2^{1014}+2^4\cdot 2^{1014}+2^6\cdot 2^{1013}+2^7\cdot 2^{1010}+8(2^8\cdot 2^{1003})$$
$$< 2^{1020}. \qquad \square$$

Solution 3.
(solution by Tony Wang)

Let a *generating set* be a set G of subsets of S such that no element $g \in G$ is the union of two or more other elements in G. For each generating set G, it trivially generates a unique union-closed set T, which is defined as the intersection of all union-closed sets which contain G (or equivalently, the smallest union-closed set which contains G). We claim that each union-closed set T also contains a unique generating set G: call an element $t \in T$ *reducible* if it can be expressed as the union of two or more other elements in T, and call it *irreducible* otherwise. On one hand, if the generating set does not contain some irreducible element of T, then it cannot generate T. On the other hand, since all irreducible elements of T must be in G, no reducible elements can be in G. Hence we have proved a bijection between the union-closed sets T and the generating sets G.

We now claim that for any set $A \subseteq \mathcal{P}(S)$, at most one of A and $B = \mathcal{P}(S)\backslash A$ is a generating set. Suppose otherwise. WLOG, we may assume that $S \in A$. To prevent S from becoming reducible in A, there must be some number which does not appear in any other element of A. WLOG, assume this is $1 \in S$. Then all other subsets of S containing 1 must appear in B. However, this includes $\{1,2\}$, $\{1,3\}$, and $\{1,2,3\}$, and hence B cannot be a generating set. This shows that at most half of all sets can be generating sets.

Finally, note that if $A = \{\{1\},\{2\},\{1,2\}\}$, then $\{3\},\{4\},\{3,4\} \in B$. In this case, neither A nor B are generating sets. Hence strictly less than half of all possible sets are generating sets.

Solution 4.
(solution by Harun Khan)

Let $U \subset \mathcal{P}(\mathcal{P}(S))$ be the set of all union-closed sets. Let $I \subset U$ be the set of all collections whose elements have pairwise different cardinalities (including the one set collections).

Now construct a function $f : U\backslash I \to U^c$ as follows. Take a set $A \in U\backslash I$. Consider all possible $k \in \{0,\dots,10\}$ such that the number of k element sets in A is 1. Start with the maximum k and suppose $B \in A$ such that $|B| = k$. Check whether there exist $P, Q \in A$, such that $B = P \cup Q$ and $P, Q \neq B$ then assign $f(A) = A\backslash\{B\}$ and terminate the process. Clearly $A\backslash\{B\} \in U^c$. Otherwise, check the next largest k and repeat. We claim that this process always terminates. Indeed, since $A \in U\backslash I$, there exist $P, Q \in A$ such that $P \neq Q$ and $|P| = |Q|$. Then $P \cup Q \in A$ and $|P \cup Q| > |P|$. If there is another set with $|P \cup Q|$ elements in A, then we repeat this. Since a set in A has a maximum of 10 elements, this will eventually stop and we will have a set in A that is the union of 2 non-empty sets in A that is unique in terms of its size (unique in A)

Now it is easy to see that f is injective. Suppose $f(A) = f(B)$. Further, suppose $A = f(A) \cup \{X\}$ and $B = f(A) \cup \{Y\}$. We have $X = P_1 \cup Q_1$ and $Y = P_2 \cup Q_2$. If $X \neq Y$ then $f(A)$ does not contain X, Y. So $f(A) \cup \{X\}$ doesn't contain Y. But it contains P_2, Q_2 (since these are different from Y). So $f(A) \cup \{X\}$ is not union-closed and we have a contradiction.

Now let's bound the codomain of f. Consider all sets that contain $\{1\},\{2\},\{3\}$ and do not contain $\{1,2\},\{1,3\}$. There are 2^{1019} such sets. Moreover these sets cannot be in the codomain of f since we need to add at least 2 sets in order to ensure they're union-closed. So the codomain of f is bounded above by $|U^c| - 2^{1019}$.

Lastly we can bound $|I|$ as follows

$$|I| < \prod_{i=0}^{10}\left(\binom{10}{i}+1\right) < \prod_{i=0}^{10} 2^8 = 2^{88}$$

since for every cardinality $i \in \{0, ..., 10\}$ there are $\binom{10}{i}$ choices of sets of cardinality i and one additional choice of not including a set of cardinality i.

Putting this all together, we get $|U| = |U \backslash I| + |I| \leq 2^{1024} - |U| - 2^{1019} + 2^{88}$, and so

$$|U| \leq 2^{1023} - 2^{1018} + 2^{87} < 2^{1023}. \qquad \square$$

Solution 5. *(solution by contestants)*

We show that there are at most 2^{896} union-closed sets T. Firstly, we compute that if $S = \{1, 2, 3\}$, then there are exactly 122 union-closed sets T.

- Suppose that T does not contain $\{1, 2, 3\}$. We note that the following T are union-closed:
    - If the union of all elements in T is exactly $\varnothing$. Then we have 2 choices: we can choose to include or exclude the empty set.
    - If the union of all elements in T is exactly one of $\{1\}$, $\{2\}$, or $\{3\}$. Then we have $3 \times 2 = 6$ choices: for each choice of $A = 1$, 2, or 3, we must include exactly $\{A\}$, and we can choose to include or exclude the empty set.
    - If the union of all elements in T is exactly $\{1, 2\}$, $\{2, 3\}$, or $\{3, 1\}$. Then we have $3 \times 4 \times 2 = 24$ choices: for each choice of $(A, B) = (1, 2)$, $(2, 3)$ or $(3, 1)$, we can have $\{\{AB\}\}$, $\{\{AB\}, \{A\}\}$, $\{\{AB\}, \{B\}\}$, $\{\{AB\}, \{A\}, \{B\}\}$, and then we can choose to include or exclude the empty set. (We could also have $\{\{A\}, \{B\}\}$, but note that as this set does not include $\{A, B\}$, it is not union-closed.)
    - If the union of all elements in T is $\{1, 2, 3\}$. This is a contradiction, and hence there are no union-closed T in this case.

  We conclude that exactly 32 out of 128 possible T which do not contain $\{1, 2, 3\}$ are union-closed.

- Suppose that T does contain $\{1, 2, 3\}$. Then the following are union-closed:
    - If T contains none of $\{1\}$, $\{2\}$, or $\{3\}$, then we have $2^4 = 16$ choices: we can choose to include or exclude $\{1, 2\}$, $\{2, 3\}$, $\{3, 1\}$, and $\varnothing$.
    - If T contains exactly one of $\{1\}$, $\{2\}$, or $\{3\}$, then we have $3 \times 16 = 48$ choices: for each choice of $\{1\}$, $\{2\}$, or $\{3\}$, we can choose to include or exclude $\{1, 2\}$, $\{2, 3\}$, $\{3, 1\}$, and $\varnothing$.
    - If T contains exactly two of $\{1\}$, $\{2\}$, $\{3\}$, then we have $3 \times 8 = 24$ choices: for each choice of $(A, B, C) = (1, 2, 3)$, $(2, 3, 1)$ or $(3, 1, 2)$, including $\{A\}$ and $\{B\}$ means we must include $\{A, B\}$, but we can still choose to include or exclude $\{A, C\}$, $\{B, C\}$, and $\varnothing$.
    - If T contains all of $\{1\}$, $\{2\}$, $\{3\}$, then it must also contain $\{1, 2\}$, $\{2, 3\}$, and $\{3, 1\}$. Hence we have two choices: we can choose to include or exclude $\varnothing$.

  Hence in this case we see that there are $16 + 48 + 24 + 2 = 90$ union-closed sets.

We have shown that there are exactly 122 union-closed sets when $S = \{1, 2, 3\}$. We now return to the case where $S = \{1, 2, 3, \ldots, 10\}$. For any union-closed set T, partition T into T_A over all subsets $A \subseteq \{4, 5, 6, 7, 8, 9, 10\}$, where $T_A = \{R \in T : A = R \cup \{4, 5, 6, 7, 8, 9, 10\}\}$. There are 128 partitions, and each partition has at most 8 elements, which differ only in the combinations of 1, 2, and 3 they include. For T to be union-closed, each T_A must be union-closed for all A. Since there are 122 possible union-closed configurations for each T_A, and there are 128 partitions, there must be at most $122^{128} < 128^{128} = 2^{896}$ union-closed T. $\square$

Problem 2. *(proposed by Harun Khan)*

Let $p > 3$ be a prime number. A sequence of $p-1$ integers $a_1, a_2, \dots, a_{p-1}$ is called *wonky* if they are distinct modulo p and $a_i a_{i+2} \not\equiv a_{i+1}^2 \pmod{p}$ for all $i \in \{1, 2, \dots, p-1\}$, where $a_p = a_1$ and $a_{p+1} = a_2$. Does there always exist a wonky sequence such that

$$a_1a_2, \qquad a_1a_2 + a_2a_3, \qquad \dots, \qquad a_1a_2 + \dots + a_{p-1}a_1,$$

are all distinct modulo p?

Notes on Marking. No marks were awarded for stating the correct answer. Some constructions which appeared to work in fact only worked when 2 was a primitive root modulo p. In this case, several marks were deducted, partly depending on the generality of the remaining cases proved.

Solution 1. *(solution by Harun Khan)*

Throughout this solution all congruences are taken modulo p. Our construction will be $a_i \equiv 1/i \bmod p$. We now verify this construction works. Note that

$$a_i \equiv a_j \implies \frac{1}{i} \equiv \frac{1}{j} \implies i \equiv j \implies i = j.$$

Then if $k < p-1$,

$$\sum_{j=1}^{k} a_j a_{j+1} \equiv \sum_{j=1}^{k} \frac{1}{j(j+1)} \equiv \sum_{j=1}^{k} \frac{1}{j} - \frac{1}{j+1} \equiv 1 - \frac{1}{k+1}$$

which are all distinct since

$$1 - \frac{1}{k+1} \equiv 1 - \frac{1}{m+1} \implies \frac{1}{k+1} \equiv \frac{1}{m+1} \implies k = m$$

Moreover $\sum_{j=1}^{p-1} a_j a_{j+1} \equiv \sum_{j=1}^{p-2} a_j a_{j+1} + a_{p-1}a_1 \equiv 1 - \frac{1}{p-1} - 1 \equiv 1$. And $1 - \frac{1}{k+1} \not\equiv 1 \ \forall k$ so this is distinct from all the previous terms. Finally if $i < p-2$

$$a_i a_{i+2} \equiv \frac{1}{i(i+2)} \not\equiv \frac{1}{i(i+2)+1} \equiv \frac{1}{(i+1)^2} \equiv a_{i+1}^2$$

Additionally if $i \in \{p-2, p-1\}$, $a_i a_{i+2} \equiv \frac{-1}{2} \not\equiv 1$ since $p \neq 3$. $\square$

Solution 2. *(solution by contestants)*

Throughout this solution all congruences are taken modulo p. Let g be a primitive root of p. Let $\ell = \frac{p-1}{2}$. Let

$$\begin{array}{llll} a_2 = g & a_4 = g^2 & \cdots & a_{p-1} = g^{\ell}, \\ a_3 = g^{\ell+1} & a_5 = g^{\ell+2} & \cdots & a_p = g^{2\ell}. \end{array}$$

First of all, the a_i's are all distinct since they are distinct powers of g. Next, we check that the wonky condition holds. Note $a_1a_3 \equiv g^{\ell+1} \not\equiv g^2 \equiv a_2^2$. Now checking for all other even and odd cases respectively,

$$\begin{aligned} a_{2r}a_{2r+2} \equiv g^{2r+1} \not\equiv g^{2r} \equiv g^{2\ell+2r} \equiv a_{2r+1}^2, \\ a_{2r+1}a_{2r+3} \equiv g^{2\ell+2r+1} \equiv g^{2r+1} \not\equiv g^{2r+2} \equiv a_{2r+2}^2. \end{aligned}$$

Finally, note that for all $1 \leq j \leq p-1$,

$$\begin{aligned} \sum_{i=1}^{j} a_i a_{i+1} &\equiv g + g^{\ell+2} + g^{\ell+3} + \cdots + g^{\ell+j} \\ &\equiv g + g^{\ell+2} \cdot \frac{g^{j-1}-1}{g-1}, \end{aligned}$$

and hence,

$$\begin{aligned} \sum_{i=1}^{j} a_i a_{i+1} \equiv \sum_{i=1}^{k} a_i a_{i+1} &\iff g + g^{\ell+2} \cdot \frac{g^{j-1}-1}{g-1} \equiv g + g^{\ell+2} \cdot \frac{g^{k-1}-1}{g-1} \\ &\iff g^{k-1} \equiv g^{j-1} \\ &\iff k = j \end{aligned}$$

$\square$

Problem 3.

(proposed by Cristi Calin)

Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be continuous functions and X be a random variable such that $E(g(X)h(X)) = 0$ and $E(g(X)^2) \neq 0 \neq E(h(X)^2)$. Prove that

$$E(f(X)^2) \geq \frac{E(f(X)g(X))^2}{E(g(X)^2)} + \frac{E(f(X)h(X))^2}{E(h(X)^2)}.$$

You may assume that all expected values exist.

Solution 1.

(solution by the ICMC Problem Committee)

As the expected value is an inner product, we can restate the problem in terms of linear algebra: What we have to prove is that, for vectors f, g, h over a vector field V such that $(g, h) = 0$ and $(g, g) \neq 0$, we have the following inequality:

$$||f||^2 \geq \frac{(f, g)^2}{||g||^2} + \frac{(f, h)^2}{||h||^2} \tag{1}$$

Let $y = f - \lambda g - \mu h$ and note that $(y, y) \geq 0$. Through the linearity of expectation and the fact that $(g, h) = 0$, we obtain that

$$0 \leq (y, y) = (f, f) + \lambda^2(g, g) + \mu^2(h, h) - 2\lambda(f, g) - 2\mu(f, h), \quad \forall \lambda, \mu \in \mathbb{R}. \tag{2}$$

Since the equation above works for any λ, μ, we can take $\lambda = \frac{(f,g)}{(g,g)}$ and $\mu = \frac{(f,h)}{(h,h)}$ to finish. □

Solution 2.

(solution by Cristi Calin)

We consider vectors again. For any $\lambda \in \mathbb{R}$, we have $\lambda(f, g) = (g, h + \lambda f) \leq ||g|| \cdot ||h + \lambda f||$, by Cauchy. Squaring, we obtain that $\lambda^2 (f, g)^2 \leq ||g||^2 \cdot ||h + \lambda f||^2$. Now dividing by $||g||^2 \neq 0$ yields

$$\lambda^2 \frac{(f, g)^2}{||g||^2} \leq ||h + \lambda f||^2 = ||h||^2 + 2\lambda(f, h) + \lambda^2 ||f||^2.$$

We can take everything on the right side, to obtain that

$$\lambda^2 (||f||^2 - \frac{(f, g)^2}{||g||^2}) + 2\lambda(f, h) + ||h||^2 \geq 0, \tag{3}$$

for all $\lambda \in \mathbb{R}$. We consider this a quadratic equation in λ, and use the fact that, for the equation to always be positive, we need the discriminant to be smaller or equal to 0. We conlcude that $||f||^2 - \frac{(f,h)^2}{||h||^2} \geq \frac{(f,g)^2}{||g||^2}$, which solves the problem. □

The proofs work for any inner product space, so instead of writing the inner product, we can apply the expected value from probability, or the integral from 0 to 2π, and the proof remains exactly the same. This problem was also proved in the Lean theorem prover – you can find a link to the proof here.

Problem 4.

(proposed by Tony Wang)

Let $\mathbb{R}^2$ denote the Euclidean plane. A continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$ maps circles to circles. (A point is not a circle.) Prove that it maps lines to lines.

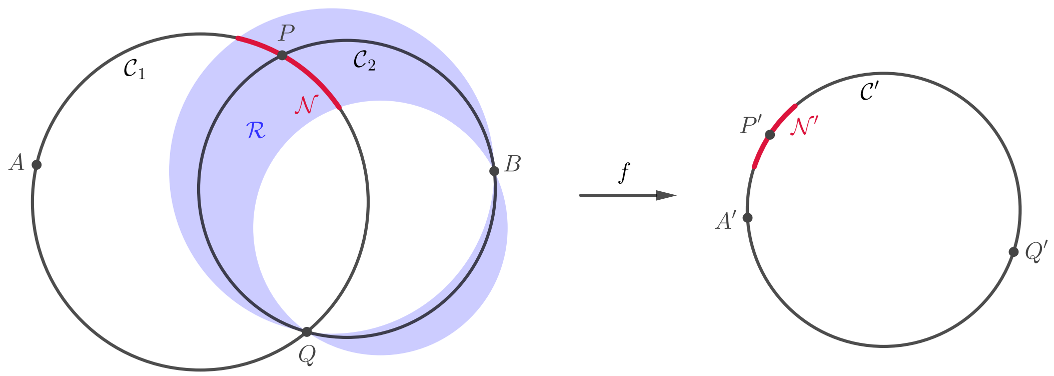
Notes on Marking. A common mistake was to assume that it suffices to prove that f preserves collinearity. This does not prove that f maps lines to lines, but 6 marks were awarded for proving this without proving that f was injective or surjective. Surprisingly, there do not appear to be any solutions based on an inversion or even a stereographic projection, so no marks were awarded for considering these. Proofs that lines mapped onto subsets of either circles or lines (worth 2 marks) was done with varying degrees of rigour. The markers felt that the rigour was necessary, so 1 mark was deducted for vagueness.

Solution 1.

(solution by Tony Wang)

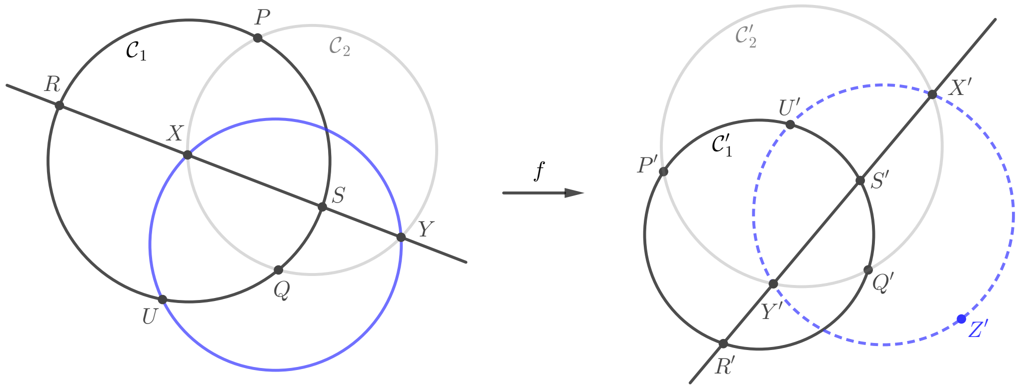
In this proof, we use the notation $X' := f(X)$. The proof can be broken down into four main steps.

- **f is injective:** Suppose that A and B are distinct points such that $A' = B'$. Since the range of the function is at least a circle, which is an uncountable set, there must exist points P and Q so that A, B, P, Q lie in general position, and A', P', Q' are distinct. Let $\mathcal{C}_1$ be the circle passing through APQ and $\mathcal{C}_2$ be the circle passing through BPQ. Note that $\mathcal{C}_1'$ and $\mathcal{C}_2'$ both map to the circumcircle $\mathcal{C}'$ of $A'P'Q'$, which must exist.



  Now, since f is continuous, there exists a neighbourhood $\mathcal{N}$ of P on $\mathcal{C}_1$ which maps to a subset of $\mathcal{C}' \backslash \{A', Q'\}$. Consider a perturbation of the circle $\mathcal{C}_2$, such that it still passes through B and Q, but now passes through a point in $\mathcal{N}$ distinct from P. Each of these circles in the perturbation must then also map to $\mathcal{C}'$ by the same argument as above. Hence, this perturbation creates a region $\mathcal{R}$ such that $f(\mathcal{R}) = \mathcal{C}'$. However, note that any circle contained entirely within $\mathcal{R}$ must then also map to exactly $\mathcal{C}'$. This implies that there are arbitrarily small circles which map to $\mathcal{C}'$, contradicting the continuity of f. Hence f is injective.

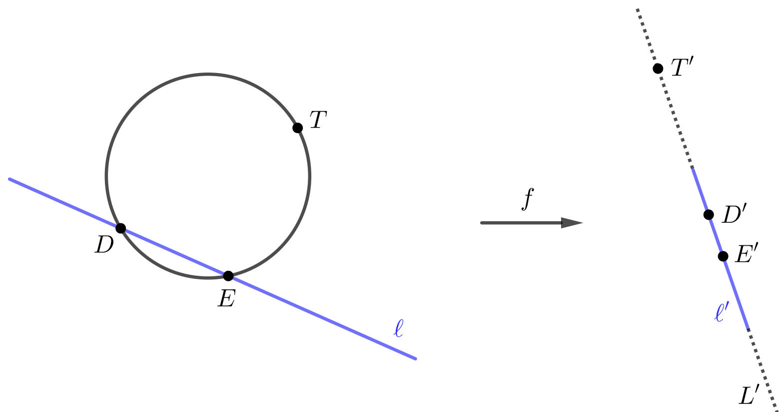
- **f is surjective:** Since f is injective, we now know that $\mathcal{C}_1'$ and $\mathcal{C}_2'$ must intersect only at P' and Q'. Let $X, Y \in \mathcal{C}_2$ so that $\mathcal{C}_1$ separates X from Y. By injective continuity we can deduce that $\mathcal{C}_1'$ separates arc $P'X'Q'$ from arc $P'Y'Q'$, and hence X' from Y'. Let the intersection of $X'Y'$ with $\mathcal{C}_1'$ be R' and S'. WLOG, assume that S is not an intersection of XY with $\mathcal{C}_1$, then the circumcircle of XYS must map to the circumcircle of $X'Y'S'$, a contradiction. Hence R and S must be the intersection of XY with $\mathcal{C}_1$.



Consider any point Z' not on the line $X'Y'$. Let the circumcircle of $X'Y'Z'$ intersect $\mathcal{C}_1'$ at U'. We know this intersection point exists because the circumcircle of $X'Y'Z'$ passes through X' and Y', which are separated by $\mathcal{C}_1'$. Since f is injective, U is distinct from S and R, and hence the circumcircle of XYU which must map to the circumcircle of $X'Y'U'$, which passes through Z'. Hence any point Z' not on the line $X'Y'$ is in the image of f. Repeating this argument replacing X with a different point W on the arc PXQ shows that every point is in the image of f. This proves that f is surjective and thus bijective.

- f **preserves collinearity:** Above, we proved that the family of circles passing through X' and Y' is mapped to by some circle passing through X and Y. Conversely, any circle passing through X and Y must map to a circle passing through X' and Y'. Hence, the family of circles passing through X and Y maps to the family of circles passing through X' and Y'. Since f is bijective the complement of the family must map to the complement of the image family. This shows that for any point T on the line XY, T' must lie on the line $X'Y'$.

- f **maps lines to lines:** Finally, since f is bijective and preserves collinearity, a line ℓ must map to a subset ℓ' of a line L'. Let $T' \in L'$. If $T' \notin \ell'$, then T must lie off the lie ℓ. Consider a circle passing through T and two points $D, E \in \ell$. This must map to a circle, but $T'D'E'$ forms a line, a contradiction. Hence, lines must map to lines. $\square$



Additionally, note that as f is bijective, parallel lines map to parallel lines. Some more analysis (or some theory) shows that f must be an affine map, but the only affine maps that map circles to circles are the maps generated by isometries and dilations. These maps also clearly work, and hence we have completely characterised f.