IMPERIAL COLLEGE
MATHEMATICS
COMPETITION

ROUND ONE

Sunday, 28 November 2021

Name: ______________________________

Contestant Number: ______________________________

University: ______________________________

Instructions:

- Do not turn over until told to do so.
- You will have 3 hours to solve 6 problems, each of which carries 10 marks.
- Use a black or blue pen or a dark pencil. Rulers, compasses, protractors, and erasers may be used but will not be required. All electronic devices, including calculators, are prohibited.
- Drinks are allowed, but food is prohibited.
- Write your solution to each problem on a different page. Write down the question number, your initials and contestant number. Use both sides whenever possible.
- Problems are listed roughly in order of difficulty. Proofs are expected for all problems even if they only ask for an answer.
- One complete solution will be awarded more marks than several partial solutions.
- Do not take away the problems sheet or any rough work when leaving the venue.

Declaration:

I hereby declare that the work I am submitting is entirely my own, and that I will not discuss or disclose the contents of this paper until 28 November 2021, 21:00 GMT.

Signature: ______________________________

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Problem	1	2	3	4	5	6	Total
Score							

Problem 1. Let T_n be the number of non-congruent triangles with positive area and integer side lengths summing to n. Prove that $T_{2022} = T_{2019}$.

Problem 2. Find all integers n for which there exists a table with n rows, 2022 columns, and integer entries, such that subtracting any two rows entry-wise leaves every remainder modulo 2022.

Problem 3. Let $\mathcal{M}$ be the set of $n \times n$ matrices with integer entries. Find all $A \in \mathcal{M}$ such that $\det(A + B) + \det(B)$ is even for all $B \in \mathcal{M}$.

Problem 4. Let p be a prime number. Find all subsets $S \subseteq \mathbb{Z}/p\mathbb{Z}$ such that

- if $a, b \in S$, then $ab \in S$, and
- there exists an $r \in S$ such that for all $a \in S$, we have $r - a \in S \cup \{0\}$.

Note: $\mathbb{Z}/p\mathbb{Z}$ denotes the integers modulo p.

Problem 5. A *tanned vector* is a nonzero vector in $\mathbb{R}^3$ with integer entries. Prove that any tanned vector of length at most 2021 is perpendicular to a tanned vector of length at most 100.

Problem 6. Is it possible to cover a circle of area 1 with finitely many equilateral triangles whose areas sum to 1.01, all pointing in the same direction?