IMPERIAL COLLEGE
MATHEMATICS
COMPETITION

ROUND TWO

Sunday, 27 February 2022

Name: ____________________

Contestant Number: ____________________

University: ____________________

Instructions:

- Do not turn over until told to do so.
- You will have 4 hours to solve 5 problems, each of which carries 10 marks.
- Use a black or blue pen or a dark pencil. Rulers, compasses, protractors, and erasers may be used but will not be required. All electronic devices, including calculators, are prohibited.
- Drinks are allowed, but food is prohibited.
- Write your solution to each problem on a different page. Write down the question number, your initials and contestant number. Use both sides whenever possible.
- Problems are listed roughly in order of difficulty. Proofs are expected for all problems even if they only ask for an answer.
- One complete solution will be awarded more marks than several partial solutions.
- Do not take away the problems sheet or any rough work when leaving the venue.

FOR MARKERS' USE ONLY						
Problem	1	2	3	4	5	Total
Score						

Problem 1. Let S be a set of 2022 lines in the plane, no two parallel, no three concurrent. S divides the plane into finite regions and infinite regions. Is it possible for all the finite regions to have integer area?

Problem 2. Evaluate

$$\frac{1/2}{1+\sqrt{2}}+\frac{1/4}{1+\sqrt[4]{2}}+\frac{1/8}{1+\sqrt[8]{2}}+\frac{1/16}{1+\sqrt[16]{2}}+\cdots$$

Problem 3. A set of points has *point symmetry* if a reflection in some point maps the set to itself. Let $\mathcal{P}$ be a solid convex polyhedron whose orthogonal projections onto any plane have point symmetry. Prove that $\mathcal{P}$ has point symmetry.

Problem 4. Fix a set of integers S. An integer is *clean* if it is the sum of distinct elements of S in exactly one way, and *dirty* otherwise. Prove that the set of dirty numbers is either empty or infinite.

Note: We consider the empty sum to equal 0.

Problem 5. A robot on the number line starts at 1. During the first minute, the robot writes down the number 1. Each minute thereafter, it moves by one, either left or right, with equal probability. It then multiplies the last number it wrote by n/t, where n is the number it just moved to, and t is the number of minutes elapsed. It then writes this number down. For example, if the robot moves right during the second minute, it would write down $2/2 = 1$.

Find the expected sum of all numbers it writes down, given that it is finite.