IMPERIAL-CAMBRIDGE
MATHEMATICS
COMPETITION

ROUND ONE

26–27 November 2022

Sponsored by DRW

Name: ____________________

Contestant Number: ____________________

University: ____________________

Instructions:

- Do not turn over until told to do so.
- You will have 3 hours to solve 6 problems, each of which carries 10 marks.
- Use a black or blue pen or a dark pencil. Rulers, compasses, protractors, and erasers may be used but will not be required. All electronic devices, including calculators, are prohibited.
- Drinks are allowed, but food is prohibited.
- Write your solution to each problem on a different page. At the top of each page, write down the question number, your initials, and your contestant number. Use both sides whenever possible. Writing clearly and not too faintly – your work will be scanned for marking.
- Problems are listed roughly in order of difficulty. Proofs are expected for all problems even if they only ask for an answer.
- One complete solution will be awarded more marks than several unfinished attempts.
- Do not take away the problems sheet or any rough work when leaving the venue.

Declaration:

I hereby declare that the work I am submitting is entirely my own, and that I will not discuss or disclose the contents of this paper until 23:00, 27 November 2022, GMT Time.

Signature: ____________________

Problem 1. Two straight lines divide a square of side length 1 into four regions. Show that at least one of the regions has a perimeter greater than or equal to 2.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(x) > f(x) > 0$ for all real numbers x. Show that $f(8) > 2022f(0)$.

Problem 3. Bugs Bunny plays a game in the Euclidean plane. At the n-th minute ($n \geq 1$), Bugs Bunny hops a distance of F_n in the North, South, East, or West direction, where F_n is the n-th Fibonacci number (defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$). If the first two hops were perpendicular, prove that Bugs Bunny can never return to where he started.

Problem 4. Let G be a simple graph with n vertices and m edges such that no two cycles share an edge. Prove that $2m < 3n$.

Note: A *simple graph* is a graph with at most one edge between any two vertices and no edges from any vertex to itself. A *cycle* is a sequence of distinct vertices $v_1, \ldots, v_n$ such that there is an edge between any two consecutive vertices, and between v_n and v_1.

Problem 5. Let $[0, 1]$ be the set $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$. Does there exist a continuous function $g : [0, 1] \to [0, 1]$ such that no line intersects the graph of g infinitely many times, but for any positive integer n there is a line intersecting g more than n times?

Problem 6. Consider the sequence defined by $a_1 = 2022$ and $a_{n+1} = a_n + e^{-a_n}$ for $n \geq 1$. Prove that there exists a positive real number r for which the sequence

$$\{ra_1\}, \{ra_{10}\}, \{ra_{100}\}, \ldots$$

converges.

Note: $\{x\} = x - \lfloor x \rfloor$ denotes the part of x after the decimal point.