IMPERIAL-CAMBRIDGE MATHEMATICS COMPETITION

6th Edition (2022–2023)

ROUND ONE

Official Solutions*

Last updated: 3 Jan 2023, 7pm

*A solution may receive full or partial marks even if it does not appear in this booklet.

Problem 1. *(proposed by Dylan Toh)*

Two straight lines divide a square of side length 1 into four regions. Show that at least one of the regions has a perimeter greater than or equal to 2.

Notes on Marking. In Solution 1, 6 marks were assigned to the case when each side of the square is intersected by one of the diving lines (case 1), and 4 marks were assigned to the other case (case 2). However, when only one case was proven, and there was no evidence that the contestant was aware of the other case, one mark was deducted. Unfortunately, this resulted in many contestants being awarded 5 marks. In some cases, either due to diagram dependency or neglect, certain cases or subcases were not addressed. (This included the case where the two dividing lines both intersected the same two adjacent sides, and the case where the two dividing lines intersected the same two opposite sides.) Where a subcase was missing, one mark was deducted.

While the isoperimetric inequality could be assumed without proof, the lemma that a triangle or square with fixed area attains minimum perimeter when regular could not be assumed (indeed, it is not the isoperimetric inequality.) Where the lemma was used to prove case 1, 2 marks were awarded for proving the triangular or quadrilateral sublemma, or 4 marks for the entire lemma, and 2 further marks were awarded for a proof of case 1 assuming the lemma. Contestants who stated that there was a region with area greater than 1/4 (or equivalent) were awarded 1 mark.

The markers are not aware of any way to prove that the "optimum" case occurs when the perimeters or areas of all regions are equal, when the intersection is at the center of the square, when the two lines are perpendicular, or indeed when the two lines are axis-aligned and intersect at the center of the square, so no marks were awarded for stating any of these. The markers are also not aware of any solution deriving from perturbations (translations or rotations of lines) from the equality case.

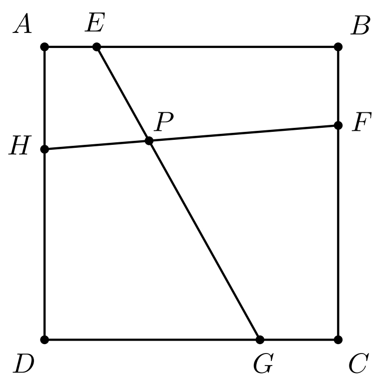
Solution 1. *(solution by Dylan Toh)*

Let $ABCD$ be the unit square. If some side (WLOG side AB) is not intersected by any line, then it is the side of some polygonal region $\mathcal{R} = ABP_1 \dots P_k$. By triangle inequality,

$$|AP_1| + |P_1P_2| + \cdots + |P_{k-1}P_k| + |P_kB| \geq |AB| = 1,$$

thus the perimeter of $\mathcal{R}$ is $\geq 2 \cdot |AB| = 2$.

Otherwise, each side AB, BC, CD, DA of the square is intersected at some point E, F, G, H respectively.



As four regions are obtained, the only possibility is for the dividing lines to be EG and FH, and they meet at some point P within the square. The sum of perimeters of all four regions is thus

$$|AB| + |BC| + |CD| + |DA| + 2 \cdot |EG| + 2 \cdot |FH| = 4 + 2 \cdot (|EG| + |FH|) \geq 4 + 2 \cdot 2 = 8,$$

as opposite sides of the square are distance 1 apart. By pigeonhole principle, some region has perimeter at least 2. □

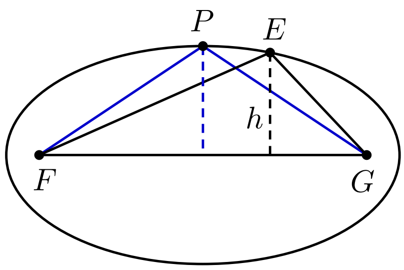
Comment. A similar proof is to split the cases based on whether or not both line segments dividing the square have a length of greater than one. If not, then it can be shown that there is one side that is not intersected.

Solution 2.

(solution by Tony Wang)

Let $ABCD$ be the unit square. Since it is divided into four regions, then at least one region, say $\mathcal{R}$, must have an area of greater than or equal to $1/4$. We now split into cases depending on how many sides $\mathcal{R}$ has.

1. $\mathcal{R}$ **has** 3 **sides.** In this case we wish to show that the triangle with a fixed area has smallest perimeter when it is equilateral. This is equivalent to showing that a triangle with a fixed perimeter has largest area when it is equilateral. Now suppose that some two sides FE and EG of the triangle are of different lengths. The locus of points E for constant $FE + EG$ is an ellipse centered on F and G passing through E.

   Since the area of triangle EFG is $\frac{1}{2}FG \times h$, where h is the height of the triangle, it is maximised when E is the apex P of the ellipse. Note that at this point, $FP = PG$. This means that a triangle with a fixed perimeter does not have the largest possible area if any two of its sides have different lengths, and therefore an equilateral triangle maximises area for a given perimeter.

   For any equilateral triangle of side length b, its height is $h = \sqrt{b^2 - (b/2)^2} = \sqrt{3}b/2$ by Pythagoras' theorem, and so its area $A = bh/2 = \sqrt{3}b^2/4$. Hence if $A \geq 1/4$, then $3b \geq 3/\sqrt[4]{3} > 3/1.5 = 2$, as desired.

2. $\mathcal{R}$ **has** 4 **sides.** In this case we wish to show that the quadrilateral with a fixed area has smallest perimeter when it is a square. To do this, first note that by a similar ellipse argument as above, a quadrilateral with fixed perimeter does not have maximal area if any two adjacent sides have differing lengths. Hence, we need only consider rhombi. Since a rhombus with side length b and angle θ has area $b^2 \sin(\theta)$, it follows that area is maximised when $\theta = 90°$, i.e. in the case of a square. Finally, any square with area greater than or equal to $1/4$ certainly has perimeter greater than or equal to 2.

3. $\mathcal{R}$ **has more than** 4 **sides.** Since there are only two dividing lines, at most two sides of $\mathcal{R}$ are contributed by the dividing lines, meaning that three or more sides are contributed by the edges of the square. Since these sides must be consecutive, one of the sides of $\mathcal{R}$ must be one of the sides of the square, say AB. Then AB contributes a perimeter of 1 to $\mathcal{R}$, and by triangle inequality, the path from B back to A contributes a length of greater than 1 to the perimeter. Hence we are done. $\square$

Solution variant (by contestants and Tony Wang). We use trigonometry to prove the that a triangle and quadrilateral with fixed area has minimal perimeter when it is regular.

1. $\mathcal{R}$ **has** 3 **sides.** Denote the three sides of the triangle by x, y, and z, and let X, Y, and Z be their opposite angles, respectively. The area of of the triangle is $\frac{1}{2}yz\sin X \geq \frac{1}{4}$ and so $\sin X \geq \frac{1}{2bc}$. Hence by sine rule we obtain

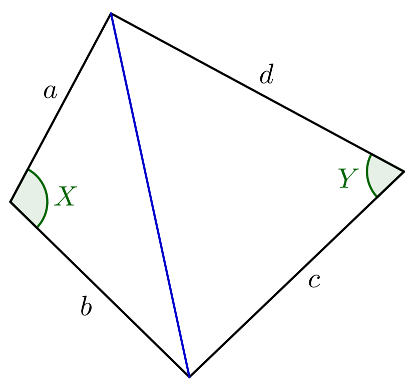
$$2xyz \geq \frac{x}{\sin X} = \frac{y}{\sin Y} = \frac{z}{\sin Z}.$$

   Rearranging, we get that

$$z \geq \frac{1}{2x\sin Y} \geq \frac{1}{2x} \qquad \text{and} \qquad y \geq \frac{1}{2x\sin Z} \geq \frac{1}{2x},$$

   and so $x+y+z \geq x + \frac{1}{x} \geq 2$ by AM-GM.

2. $\mathcal{R}$ **has** 4 **sides.** Denote the four sides of the quadrilateral by a, b, c, and d, in order, and let the angle between a and b be X, and the angle between c and d be Y.



   Then we have

$$\frac{1}{2}ab\sin X + \frac{1}{2}cd\sin Y \geq \frac{1}{4}. \tag{1}$$

   By cosine rule, we have $a^2+b^2-2ab\sin X = c^2+d^2-2cd\sin Y$, which implies

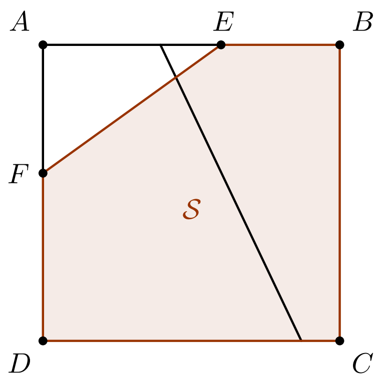
$$\frac{1}{2}ab\sin X = \frac{a^2+b^2-c^2-d^2}{4} + \frac{1}{2}cd\sin Y,$$

   and so substituting into (1) and multiplying by 4 yields $a^2+b^2-c^2-d^2+4cd\sin Y = 4 \times \text{Area} \geq 1$. Returning now to our goal of minimising $a+b+c+d$, we note that if $a \neq b$, then by replacing both a and b by $\sqrt{(a^2+b^2)/2}$, the area of $\mathcal{R}$ is preserved but the perimeter is decreased, since a short calculation will show that $2\sqrt{(a^2+b^2)/2} \geq a+b$ reduces to $(a-b)^2 \geq 0$. Hence the minimum perimeter is achieved only if $a = b$. Similarly, we can deduce that $b = c$ and $c = d$. So it remains to minimise $4a$ when $4a^2\sin Y \geq 1$, but then we have $a^2 \geq \frac{1}{4} \implies a \geq \frac{1}{2} \implies 4a \geq 2$, as required. $\square$

Comment. There are many other ways of proving the isoperimetric inequality for triangles and quadrilaterals, including by use of calculus.

Solution 3. *(solution by contestants)*

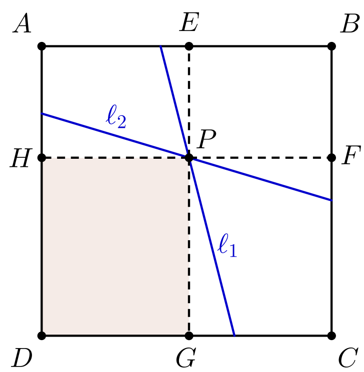
Let $ABCD$ be the unit square. If both line segments dividing the square have length greater than or equal to one, then we use the pigeonhole argument in the second half of Solution 1. Otherwise, one of the lines has length less than one, and so it must intersect two adjacent sides. WLOG, we can assume it intersects sides AB and AD at E and F respectively. Let x be the length of AE and y be the length of AF. Since EF has length less than one, we have $\sqrt{x^2+y^2} < 1$. After squaring, adding $-2xy$ to both sides, and factorising, it becomes $(x-y)^2 < 1-2xy$, and since $(x-y)^2 \geq 0$, we get $xy < 1/2$. Now we know that AEF has an area of $xy/2 < 1/4$, and so the region $\mathcal{S} = EBCDF$ has area greater than $3/4$.



Now, the second line must divide $\mathcal{S}$ into two regions, and by pigeonhole principle at least one of these regions will have an area greater than $3/8$. By the isoperimetric inequality, any region with area greater than $3/8$ will have greater perimeter than the circle of the same area. But if $\pi r^2 > 3/8$ then $2\pi r > \sqrt{3\pi/2} > \sqrt{4} = 2$, and so the region with area greater than $3/8$ will have perimeter greater than 2, as required. $\square$

Solution 4. *(solution by contestants)*

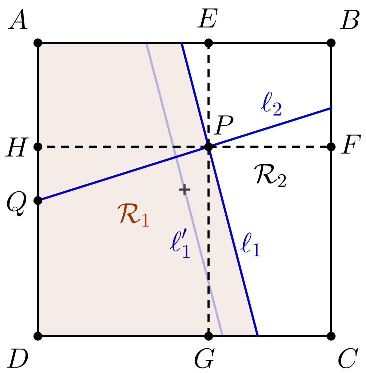
Let $ABCD$ be the unit square, ℓ_1 and ℓ_2 be the two dividing lines, and P their point of intersection. Draw altitudes from P to AB, BC, CD, and DA and let their feet be E, F, G, and H respectively. The lines EG and FH divide the square into four rectangles.



Let $\mathcal{P}(\mathcal{R})$ denote the perimeter of region $\mathcal{R}$. We now consider two cases:

1. **ℓ_1 and ℓ_2 combined intersect the interiors of only zero or two rectangles.** In this case there must be a pair of two rectangles touching only at a vertex which are not intersected by ℓ_1 or ℓ_2: suppose they are $PEBF$ and $PGDH$. Then note that the $\mathcal{P}(PEBF) + \mathcal{P}(PGDH) = 2(PE + PG + PF + PH) = 2(1 + 1) = 4$, and so at least one of them must have a perimeter greater than or equal to two: suppose $PGDH$ does. Then since $PGDH$ is contained inside another region, that region must have perimeter greater than 2, as required.[†]

2. **ℓ_1 and ℓ_2 combined intersect the interiors of all four rectangles.** In this case, suppose WLOG that the region $PGDH$ contains the center of the square (not necessarily in its interior). Then we know that $PG, PH \geq 1/2$. WLOG we may assume that ℓ_1 is the line that passes through regions $PHAE$ and $PFCG$. Denote the two regions that the line ℓ_1 divides the square into by $\mathcal{R}_1$ and $\mathcal{R}_2$.



Construct a line ℓ_1' parallel to ℓ_1 which passes through the center of the square, dividing the square into two congruent regions $\mathcal{R}_1'$ and $\mathcal{R}_2'$. Note that ℓ_1' has a length x between 1 and $\sqrt{2}$, and so the $\mathcal{P}(\mathcal{R}_1') + \mathcal{P}(\mathcal{R}_2') = 4 + 2x \geq 6$, which implies that $\mathcal{P}(\mathcal{R}_1'), \mathcal{P}(\mathcal{R}_2') \geq 3$. Since ℓ_1 either does not intersect ℓ_1' or is the same line as ℓ_1', one of $\mathcal{R}_1$ and $\mathcal{R}_2$ must contain one of the regions $\mathcal{R}_1'$ and $\mathcal{R}_2'$. WLOG we may suppose $\mathcal{R}_1$ contains $\mathcal{R}_1'$. Then by a similar argument to above, we have $\mathcal{P}(\mathcal{R}_1) \geq \mathcal{P}(\mathcal{R}_1') \geq 3$.

Now, ℓ_2 must pass through P and intersect either DG or DH. Call this intersection point Q. By triangle inequality we have either $PQ \geq PG$ or $PQ \geq PH$ depending on whether Q is on DG or DH, but in either case we can conclude that $PQ \geq 1/2$. Hence the sum of perimeters of the two regions that ℓ_2 divide $\mathcal{R}_1$ into must be at least $3 + 2PQ \geq 4$, and so by pigeonhole principle at least one of the two regions must have perimeter at least 2. $\square$

[†]This may be proven by repeatedly applying triangle inequality, or by perpendicularly "offsetting" the sides of the rectangle until it reaches the containing region.

Problem 2. *(proposed by Ethan Tan)*

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(x) > f(x) > 0$ for all real numbers x. Show that $f(8) > 2022f(0)$.

Notes on Marking. Most contestants used arguments similar to the model solutions proposed. Common mistakes included arguing that any such function must be purely exponential, or not giving reasonable justification to why such a function takes values greater than the exponential function e^x on the positive real axis. One mark was deducted for neglecting to convincingly show why $e^8 > 2022$. Giving examples of functions satisfying the inequalities was not awarded any marks.

Solution 1. *(solution by Ethan Tan)*

We have

$$\frac{d}{dx} \log f = \frac{f'}{f} > 1 \implies \log f(x) > \log(f(0)) + x$$

for all $x > 0$. So $f(x) > f(0)e^x$. Since $e^2 > 2.7^2 > 7$, we have $e^8 > 7^4 = 2401 > 2022$, as required. $\square$

Solution 2. *(solution by Ethan Tan)*

We have, for all $x > 0$,

$$\frac{d}{dx}\left(e^{-x} f\right) = e^{-x}(f' - f) > 0 \implies e^{-x} f(x) > f(0).$$

So $f(x) > f(0)e^x$. We can finish as in solution 1. $\square$

Solution 3. *(solution by Ethan Tan)*

The condition $f'(x) > 0$ implies that f is strictly increasing. Hence, for all $x > y$ we have $f'(x) > f(x) > f(y) > 0$, and so there exists $z \in (y, x)$ such that $f'(z) = \frac{f(x)-f(y)}{x-y} > f(z) > f(y)$ by the mean value theorem. Therefore, $f(x) > (1+x-y)f(y)$ for all $x > 0$. Letting $y = x - 1/n$ with $n \in \mathbb{N}$ we get

$$f(x) > \left(1 + \frac{1}{n}\right) f(y) \qquad \implies \qquad f(8) > \left(1 + \frac{1}{n}\right) f\left(8 - \frac{1}{n}\right) > \cdots > \left(1 + \frac{1}{n}\right)^{8n} f(0).$$

Taking limits we find that $f(8) > f(0)e^8$ and we can finish as in solution 1. $\square$

Solution 4. *(solution by Ethan Tan and Tony Wang)*

Define $g(x) = \frac{f(x)}{f(0)} - e^x$, and note that $g'(x) > g(x)$ and $g(0) = 0$. Since $g'(0) > 0$, there exists an interval around 0 where g is positive. Let S be the set $\{x > 0 : g(x) \leq 0\}$. we will show that S is empty.

Suppose for the sake of contradiction that $S \neq \varnothing$, so there exists a positive $a = \inf S$. Then $g(0) = g(a) = 0$ and by the mean value theorem there exists $0 < b < a$ satisfying $g'(b) = 0$. However, using the properties of g, we then have $0 < g(b) < g'(b) = 0$, so $b \in S$, contradicting the minimality of a.

Hence $S = \varnothing$, i.e. $g(x) > 0$ for all $x > 0$. But this rearranges into $f(x) > f(0)e^x$ for all $x > 0$, and we can finish as in solution 1. $\square$

Comment. Using auxiliary function can be avoided by directly comparing $f(x)/f(0)$ to e^x for $x > 0$ and using a series expansion around a similarly constructed infimum. The argument using the infimum could be replaced by one showing that the set of positive numbers where g is positive is unbounded.

Problem 3.

(proposed by Dylan Toh)

Bugs Bunny plays a game in the Euclidean plane. At the n-th minute ($n \geq 1$), Bugs Bunny hops a distance of F_n in the North, South, East, or West direction, where F_n is the n-th Fibonacci number (defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$). If the first two hops were perpendicular, prove that Bugs Bunny can never return to where he started.

Notes on Marking. Almost all contestants who obtained 9-10 marks followed Solution 1 below with some combination of the alternative branching solutions listed. Some contestants using this method forgot to mention that it is not possible to return in $N = 1, 2, 3$ steps but no marks were deducted for this since this is painfully obvious.

By far the most common incorrect approach was to try show that after taking the first 4 steps $(1, 0) + (0, 1) + (0, 2) - (0, 3) = (1, 0)$ it is impossible to return to the origin. It is not immediately clear that these must be the 3rd and 4th steps and no mark were awarded for making this claim.

Very few contestants solved the problem via Solution 2, and all but one solved it using induction, which requires the lemma in Solution 1 (or similar). As a side note since Zeckendorf's theorem implicitly uses the lemma in Solution 1, all solutions by contestants/staff require some form of this identity.

Solution 1.

(solution by Dylan Toh and Tiger Ang)

First we claim that the following lemma holds.

Lemma. For all $n \geq 0$, $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$.

This may be proven by induction on n, or by noting that the sum telescopes as $F_k = F_{k+2} - F_{k+1}$. It should be noted that other similar identities/inequalities can also be used.

Let x_n, y_n denote the x and y coordinate of Bugs Bunny after time n. Now we suppose for a contradiction that Bugs Bunny returns to the origin after $N \geq 4$ steps. WLOG let the final N-th step be in the South direction, then the final position is $(x_{N-1}, y_{N-1}) = (0, F_N)$. If the $(N-1)$-th jump was not North, then

$$y_{N-2} \geq F_N = F_1 + F_2 + \cdots + F_{N-2} + 1$$

Which is a contradiction since this point cannot be reached by step $(N-2)$ even if Bugs Bunny always traveled north. Hence the second last jump can only be in the North direction and $(x_{N-2}, y_{N-2}) = (0, F_{N-2})$. If the $(N-2)$-th jump was not North then, (x_{N-3}, y_{N-3}) is one of 3 cases $\{(0, 2F_{N-2}), (F_{N-2}, F_{N-2}), (-F_{N-2}, F_{N-2})\}$. In all cases

$$|x_{N-3}| + |y_{N-3}| = 2F_{N-2} > F_{N-1} = F_1 + F_2 + \cdots + F_{N-3} + 1$$

Which is a contradiction since this point cannot be reached by step $(N-3)$ even if Bugs Bunny always traveled north or north/east or north/west respectively. Hence the third last jump can only be in the North direction and $(x_{N-3}, y_{N-3}) = (0, 0)$. Hence there is no solution in N-steps, only if there is no solution in $(N-3)$-steps. So we can apply induction with base cases $N = 1, 2, 3$ all of which are trivial to check. $\square$

Alternative branching solution: We can also show the third last jump can only be in the North direction by showing the other three directions give a contradiction similar to the previous case. If the $(N-2)$-th jump was South then

$$y_{N-3} = 2F_{N-2} \geq F_{N-1} = F_1 + F_2 + \cdots + F_{N-3} + 1$$

Which is a contradiction since this point cannot be reached by step $(N-3)$ even if Bugs Bunny always traveled North.

If the $(N-2)$-th jump was East/West then by a similar reasoning to the above it must both be true that the $(N-3)$-th jump was West/East and the $(N-3)$-th jump was North a contradiction.

Alternative branching solution: If we had previously assumed that $N \geq 4$ is the minimal such value such that Bugs Bunny returns to the origin after N-steps. Then the existence of a solution of $(N-3)$-steps gives an immediate contradiction. Note that the base cases $N = 1, 2, 3$ still need to be checked here.

Solution 2. *(solution by contestants)*

First we claim the following lemma holds.

Lemma. Suppose that $A, B \subset \{3, 4, \ldots, N\}$ such that $A \cap B = \emptyset$ and

$$1 + \sum_{a \in A} F_a = \sum_{b \in B} F_b.$$

Then $3 \in A$.

If this lemma holds we are done since there exists a solution in N-steps iff we can find a disjoint partition $X_+, X_-, Y_+, Y_- \subset \{3, 4, \ldots, N\}$ such that (WLOG $(x_2, y_2) = (1, 1)$)

$$1 + \sum_{i \in X_+} F_i = \sum_{j \in X_-} F_j \text{ and } 1 + \sum_{i \in Y_+} F_i = \sum_{j \in Y_-} F_j$$

By the lemma this is a contradiction since $3 \in X_+ \cap Y_+ = \emptyset$.

It remains to prove the lemma which can be done by induction on $m = \max(A \cup B)$. By a similar logic to Solution 1, $m \in A \Rightarrow m-1, m-2 \in B$ and $m \in B \Rightarrow m-1, m-2 \in A$. Hence there is a solution where $m = \max(A \cup B)$ only if there is a solution where $m-3 = \max(A \cup B)$. So we can apply induction with base cases $m = 3, 4, 5$ which are trivial to check. $\square$

Alternative branching solution (by Dylan Toh): We can prove the lemma by using Zeckendorf's theorem. Suppose in A we take the greatest n such that $n, n-1 \in A$ and replace A with $(A \backslash \{n, n-1\}) \cup \{n+1\}$, then the equality of the sums are preserved. Repeat this with both sets A and B until no more replacements can be made and call the new sets $\tilde{A}$ and $\tilde{B}$. This process must terminate since the cardinality of the set decreases with each replacement. In particular we have

$$S = 1 + \sum_{a \in \tilde{A}} F_a = F_2 + \sum_{a \in \tilde{A}} F_a = \sum_{b \in \tilde{B}} F_b$$

We observe that there are no consecutive integers in $\tilde{A}$ and $\tilde{B}$. If we suppose now that $3 \notin \tilde{A}$ then we have two different Zeckendorf representations of S, a contradiction. Hence $3 \in \tilde{A}$ which implies $3 \in A$.

Problem 4.

(proposed by Ethan Tan)

Let G be a simple graph with n vertices and m edges such that no two cycles share an edge. Prove that $2m < 3n$.

Note: A *simple graph* is a graph with at most one edge between any two vertices and no edges from any vertex to itself. A *cycle* is a sequence of distinct vertices $v_1, \ldots, v_n$ such that there is an edge between any two consecutive vertices, and between v_n and v_1.

Notes on Marking. Marks were most often lost for not fully justifying steps. For example, for Solution 1, candidates should give a brief explanation for why it is valid to remove edges from each cycle, and some justification for properties of trees i.e. $n > m$ (it is acceptable to say this is well-known). For Solution 4, cycle contraction needs to be justified carefully, it is least prone to error if framed in terms of finding a minimal counterexample. Many candidates tried to contract all cycles at once, and did not justify why that worked; from that point on solutions were usually awarded low or no marks. Some candidates attempted direct induction on n – this is difficult to do rigorously and attempts were often awarded 0 marks. 1 mark was deducted for not addressing the fact that multiple cycles may meet at a vertex for Solution 3, and 2 marks for not justifying the graph is planar for Solution 2. Trying to optimise the ratio of m to n "intuitively" was awarded at most 1 mark.

Solution 1.

(solution by contestants)

Every cycle contains ≥ 3 edges, but every edge is in ≤ 1 cycle. Denote the number of cycles by C, then we have $C \leq m/3$. Delete an edge from each cycle to obtain the graph G'. G' has no cycles, so it has at most $n-1$ edges. (This is a well-known fact.) Then $m = |E(G)| = C + |E(G')| \leq m/3 + n - 1 < m/3 + n$, and hence $2m/3 < n$ as required. □

Solution 2.

(solution by contestants)

G must be planar. This can be most quickly shown using Kuratowski's Theorem. Suppose G is not planar, then it contains a subgraph $K_{3,3}$ or K_5, which contradicts the assumption that cycles in G do not share an edge.

Therefore G is planar, so every cycle contributes 1 to the number of faces. Since the number of cycles C satisfies $C \leq E/3$, and $F = C+1$, applying Euler's formula $F-E+V = 2$ completes the proof. □

Solution 3.

(solution by contestants)

Define a new graph H based on cycles, and the cycles that share a vertex. In other words, let the cycles be $\{c_1, c_2, ..., c_k\}$. Then let the c's be vertices of the graph H, with an edge connecting c_i and c_j if and only if the original cycles share a vertex.

It should be observed that several cycles may meet at a vertex, which would create a complete subgraph and cause the proof to fail. Candidates should construct some method to avoid this problem, for example, identifying the edges of such a subgraph differently from other edges.

It can then be deduced that the graph H is a forest, because if H contains cycles, one can easily find cycles that share an edge. Specifically, if $c_{i_1}, c_{i_2}...c_{i_l}$ is a cycle, consider traversing these graphs along the same shared vertices, but choosing different sides of a cycle. This leads to a contradiction.

There are then several ways to complete the proof. For example, H must have a leaf node, implying the existence of a cycle which is connected to at most one other cycle, and one can induct on the number of cycles. Alternatively, one could bound the number of edges that must be removed for the graph to become acyclic. $\square$

Solution 4. *(solution by Ethan Tan)*

Suppose not; choose a counterexample G with minimal n. Choose a cycle C in G and contract it to a single vertex to obtain a new G'. Note, this is possible because no two vertices on the cycle can be joined by a path not belonging to the cycle.

No two cycles of G' share an edge (or we could un-contract G' to G and these cycles would share an edge in G), so $2|E(G')| < 3|V(G')|$ by minimality of G. But $m = |E(G')| + c$ and $n = |V(G')| + c - 1$ where c is the number of vertices in C, and so $2(m - c) < 3(n - c + 1)$, i.e. $2m < 3n - c + 3 \leq 3n$ since $c \geq 3$, a contradiction. $\square$

Solution 5. *(solution by contestants)*

If G contains an edge, e, which is not part of any cycle, then define H to be the 'shortened' graph of G where e is removed, and any edge that connects to the vertices to either side of e, say u and v, instead connect to u. This shortened graph also satisfies the condition no two cycles share an edge, because e was not part of a cycle by assumption, and this 'shortening' procedure cannot create new cycles because it would create a contradiction. It can then be shown that if H satisfies $2m < 3n$, then G also satisfies this condition, since this procedure changes both m and n by one.

Thus, it remains to be shown that once G has been contracted until all edges are in cycles, it satisfies $2m < 3n$. This is straightforward and can be done by induction on the number of cycles. $\square$

Problem 5. *(proposed by Ethan Tan)*

Let $[0,1]$ be the set $\{x \in \mathbb{R} : 0 \le x \le 1\}$. Does there exist a continuous function $g : [0,1] \to [0,1]$ such that no line intersects the graph of g infinitely many times, but for any positive integer n there is a line intersecting g more than n times?

Notes on Marking. No marks were awarded for stating the correct answer. 1 partial mark was awarded for each of the ideas of using oscillatory functions to obtain arbitrarily many intersections, and using an envelope to preserve finiteness of intersections. A valid construction without proof was awarded 4 marks. Justifying the properties of arbitrarily many intersections and finiteness of intersections was awarded an additional 3 marks each. 1 mark was deducted for using linear segments in an otherwise valid construction. 1 mark was deducted for incomplete justification of either property, or rotating graph segments without justifying that it is well-defined.

Solution 1. *(solution by Dylan Toh)*

We show that the answer is yes. The construction is a function with microscopic oscillatory behaviour within a suitable envelope.

For $n \ge 1$, let $P_n(x)$ be the $2n$-th Chebyshev polynomial, i.e. the polynomial such that $\cos(2n\theta) = P_n(\cos\theta)$ for all θ. We shall use the following properties of $P_n(x)$:

- P_n has degree $2n$.
- $|x| \le 1 \implies |P_n(x)| \le 1$.
- The extrema of P_n are attained at $-1 = x_0^{(n)} < x_1^{(n)} < \cdots < x_{2n}^{(n)} = 1$, with $P_k(x_i^{(n)}) = (-1)^k$ for $k = 0, 1, \ldots, 2n$. (Explicitly, $x_k = \cos(k\pi/2n)$.)

Consider the function $g : [0,1] \to \mathbb{R}$ given by

$$g(x) = \begin{cases} \frac{x^2+x^3}{2} + \frac{x^2-x^3}{2} P_n\left(2n(n+1)x - 2n - 1\right), & x > 0, \quad n = \lfloor x^{-1} \rfloor \\ 0, & x = 0. \end{cases}$$

First, we show g is continuous. Note g is piecewise polynomial: on each interval $x \in I_n = [(n+1)^{-1}, n^{-1}]$ for $n \ge 1$,

$$g(x) = \frac{x^2+x^3}{2} + \frac{x^2-x^3}{2} P_n\left(2n(n+1)x - 2n - 1\right)$$

with $-1 \le 2n(n+1)x - 2n - 1 \le 1$. Thus $x^2 \le g(x) \le x^3$ for all $x \in I_n$.

Since $P_n(\pm 1) = 1$, thus g is continuous between pieces. Since $x^2 \le g(x) \le x^3$ for all $x \in [0,1]$, thus by the squeeze theorem, $g(x) \to 0 = g(0)$ as $x \to 0^+$, thus g is continuous at 0. Thus g is continuous on $[0,1]$, and has image in $[0,1]$ as well.

Next, we claim that for any positive integer n, there is a line intersecting the graph of g at least n times. Consider $(n+1)^{-1} = a_0 < a_1 < \cdots < a_{2n} = n^{-1}$, where for each $k = 0, 1, \ldots, 2n$,

$$a_k = \frac{2n+1+x_k^{(n)}}{2n(n+1)}.$$

For each $k = 0, 1, \dots, 2n$, the point

$$A_k = (a_k, g(a_k)) = \left(a_k, \frac{a_k^2 + a_k^3}{2} + \frac{a_k^2 - a_k^3}{2}(-1)^k\right) = \begin{cases} (a_k, a_k^2), & k \text{ even} \\ (a_k, a_k^3), & k \text{ odd} \end{cases}$$

lies on the graph of g. Note for sufficiently large n, $a_k^3 \leq n^{-3} < 2n^{-3} < (n+1)^{-2} < a_l^2$ for all odd indices k and even indices l. Considering the line $y = 2n^{-3}$, the even-indexed points A_l lie above the line, while the odd-indexed points A_k lie below the line. By the intermediate value theorem, the graph of g intersects the line $y = 2n^{-3}$ within $a_{k-1} < x < a_k$ for each $k = 1, \dots, 2n$. This guarantees at least $2n$ intersections.

Lastly, we claim that any line intersects the graph of g at only finitely many points. Suppose otherwise, that some line ℓ intersects the graph of g at infinitely many points. For each $n \geq 1$, g is a non-linear polynomial on each I_n, thus has finitely many intersections with ℓ on I_n.

Since $[0, 1] = \{0\} \cup I_1 \cup I_2 \cup \dots$, thus there exists an increasing sequence of indices $n_1 < n_2 < n_3 < \dots$ such that ℓ intersects g at some point (α_k, β_k), where $\alpha_k \in I_{n_k}$ for each $k \geq 1$. Note $0 < \alpha_k \leq n_k^{-1}$ and $0 < \beta_k \leq \alpha_k^2$ for all $k \geq 1$. Thus in the limit $k \to \infty$, $(\alpha_k, \beta_k) \to (0, 0)$ lies on ℓ, and

$$0 \leq \frac{\beta_k}{\alpha_k} \leq \alpha_k \to 0 \quad \text{ as } k \to \infty,$$

thus ℓ has gradient 0. But $g(x) > 0$ for all $x > 0$, so ℓ only intersects the graph of g at the origin, a contradiction. The result follows. $\square$

Comment. The above construction involves envelopes $x^2 \leq g(x) \leq x^3$, with arbitrarily large number of intersections provided by the Chebyshev polynomials. Oscillatory behaviour may instead be obtained with trigonometric functions, and a wide class of envelopes may be employed to preserve the finiteness of intersections. There are also constructions for which g is analytic on $0 < x < 1$, such as $g(x) = \left(\frac{2+\sin(1/x)}{3}\right) x^2$.

Problem 6. *(proposed by Ethan Tan)*

Consider the sequence defined by $a_1 = 2022$ and $a_{n+1} = a_n + e^{-a_n}$ for $n \geq 1$. Prove that there exists a positive real number r for which the sequence

$$\{ra_1\}, \{ra_{10}\}, \{ra_{100}\}, \ldots$$

converges.

Note: $\{x\} = x - \lfloor x \rfloor$ denotes the part of x after the decimal point.

Notes on Marking. Trying to show that a_n converges was not awarded any marks (indeed, a_n does not converge).

Solution 1. *(solution by Ethan Tan)*

Note that for $0 < x < 1$, since e^x is convex, we have

$$1 + x < e^x < (e-1)x + 1 < 2x + 1 \tag{†}$$

since $1 + x$ is the tangent at 0 and $(e-1)x + 1$ is the line from $(0, e^0)$ to $(1, e^1)$. Integrating both sides of $e^x < 2x + 1$ (noting that they are equal at $x = 0$) we have $e^x < 1 + x + x^2$ on $0 < x < 1$.

Let $b_n = e^{a_n}$, so that $\ln(b_{n+1}) = \ln(b_n) + 1/b_n$. Now $a_n > e$ for all n, so $b_n > 1$. We then get that $b_{n+1} = b_n e^{1/b_n}$, so by (†), $b_n(1 + 1/b_n) < b_n e^{1/b_n} < b_n(1 + 1/b_n + 1/b_n^2)$. Hence

$$b_n + 1 < b_{n+1} < b_n + 1 + 1/b_n.$$

Applying the LHS repeatedly gives $b_n > n$ (since $b_1 > 1$). Using the RHS, we have $b_{n+1} < b_n + 1 + 1/b_n < b_n + 1 + 1/n$, so applying this repeatedly, we see that

$$\begin{aligned} b_{n+1} &< b_1 + (n-1) + (1 + 1/2 + 1/3 + \ldots + 1/n) \\ &< b_1 + (n + 1/2) + \int_2^{n+1} \frac{1}{x}\, dx \\ &= b_1 + (n + 1/2) + \ln(n+1) - \ln(2) < n + \ln(n+1) + 10 \end{aligned}$$

since $\ln(2022) < 10$.

So $n < b_n < n + \ln(n+1)$, i.e. $\ln(n) < a_n < \ln(n + \ln(n+1) + 10)$. Choose $r = \log_{10}(e)$. Then

$$\log_{10}(10^k) < ra_{10^k} < \log_{10}(10^k + \ln(10^k + 1) + 10).$$

Now $\log_{10}(10^k) = k$, whereas

$$\log_{10}(10^k + \ln(10^k + 1) + 10) = k + \log_{10}(1 + \ln(10^k + 1)/k + 10/k) = k + o(1).$$

So we have the desired result. □