IMPERIAL-CAMBRIDGE
MATHEMATICS
COMPETITION

ROUND TWO

26 February 2023

Sponsored by DRW

Name: ______________________________

Contestant Number: ______________________________

University: ______________________________

Instructions:

- Do not turn over until told to do so.
- You will have 4 hours to solve 5 problems, each of which carries 10 marks.
- Use a black or blue pen or a dark pencil. Rulers, compasses, protractors, and erasers may be used but will not be required. All electronic devices, including calculators, are prohibited.
- Drinks are allowed, but food is prohibited.
- Write your solution to each problem on a different page. At the top of each page, write down the question number, your initials, and your contestant number. Use both sides whenever possible. Writing clearly and not too faintly – your work will be scanned for marking.
- Problems are listed roughly in order of difficulty. Proofs are expected for all problems even if they only ask for an answer.
- One complete solution will be awarded more marks than several unfinished attempts.
- Do not take away the problems sheet or any rough work when leaving the venue.

Problem 1. The city of Atlantis is built on an island represented by $[-1, 1]$, with skyline initially given by $f(x) = 1 - |x|$. The sea level is currently $y = 0$, but due to global warming, it is rising at a rate of 0.01 a year. For any position $-1 < x < 1$, while the building at x is not completely submerged, then it is instantaneously being built upward at a rate of r per year, where r is the distance (along the x-axis) from this building to the nearest completely submerged building.

How long will it be until Atlantis becomes completely submerged?

Problem 2. Show that if the distance between opposite edges of a tetrahedron is at least 1, then its volume is at least $1/3$.

Problem 3. The numbers $1, 2, \ldots, n$ are written on a blackboard and then erased via the following process:

- Before any numbers are erased, a pair of numbers is chosen uniformly at random and circled.
- Each minute for the next $n - 1$ minutes, a pair of numbers still on the blackboard is chosen uniformly at random and the smaller one is erased.
- In minute n, the last number is erased.

What is the probability that the smaller circled number is erased before the larger?

Problem 4. Do there exist infinitely many positive integers m such that the sum of the positive divisors of m (including m itself) is a perfect square?

Problem 5. A clock has an hour, minute, and second hand, all of length 1. Let T be the triangle formed by the ends of these hands. A time of day is chosen uniformly at random. What is the expected value of the area of T?