IMPERIAL-CAMBRIDGE
MATHEMATICS
COMPETITION

6th Edition (2022–2023)

ROUND TWO

Official Solutions*

Last updated: 5 Apr 2023, 11am

*A solution may receive full or partial marks even if it does not appear in this booklet.

Problem 1. *(proposed by Ethan Tan)*

The city of Atlantis is built on an island represented by $[-1, 1]$, with skyline initially given by $f(x) = 1 - |x|$. The sea level is currently $y = 0$, but due to global warming, it is rising at a rate of 0.01 a year. For any position $-1 < x < 1$, while the building at x is not completely submerged, then it is instantaneously being built upward at a rate of r per year, where r is the distance (along the x-axis) from this building to the nearest completely submerged building.

How long will it be until Atlantis becomes completely submerged?

Notes on Marking. By far the most common solution type was deriving constraints based on the intersection of the island with water, which by definition has height $0.01t$ at the point of submersion. The next most common solution type involved the fact the gradient changed with rate ± 1 (depending on parameterisation). These two solution types cover almost all full-mark solutions. The nature of the problem meant candidates usually received 0–1 marks, or 9–10 marks. It was extremely rare for candidates to obtain the correct equations, but fail to solve them. Sometimes, candidates made intuitive but unjustified assumptions about the problem, for example, forgetting to state they were considering half the island, for which 1 or 2 marks may be deducted. Although a rigorous proof that the island profile is linear was not expected, and was not assumed in the most common solution type, if candidates assume this in the solution, they are expected to justify it. Several candidates interpreted time to be discrete. It appears some candidates first discretised time, then took the continuum limit, this was rarely successful.

Solution 1. *(solution by Tony Wang and Yuqing Wu)*

Parameterise time t in years, and the x-coordinate of the waterfront in the first quadrant by $a(t)$. Denote the height of the island at time t as $f_t(x)$. Note that the rate of change of a depends only on the gradient of the island at the waterfront. By inspection (or the inverse function rule), if $g_t = \frac{\mathrm{d}}{\mathrm{d}x} f_t$ is the gradient, then we can see that

$$\dot{a}(t) = \frac{0.01}{g_t(a(t))},$$

where $\dot{a}(t)$ represents the time-derivative of a.

For all non-negative x-coordinates $x_1 < x_2$, the difference in the distance from x_1 to the waterfront and x_2 to the waterfront is the constant $x_2 - x_1$ as long as neither been submerged. Hence, the difference in their rates of being built upwards will also be the constant $x_2 - x_1$, and so $f_t(x_1) - f_t(x_2) = (x_2 - x_1)(t + 1)$. By differentiation from first principles, this means that the gradient at $f_t(x)$ for all positive non-submerged x will be exactly $-t - 1$. So we have the equation

$$\dot{a}(t) = -\frac{0.01}{t + 1},$$

which yields $a(t) = 0$ at $t = e^{100} - 1$. $\square$

Solution 2. *(solution by contestants)*

Parameterise the x-coordinate of the waterfront in the first quadrant by $a(t)$, where t is the time in years. The rate of construction at any point x is, by definition,

$$\frac{\partial f(x, t)}{\partial t} = a(t) - x.$$

For the sake of simplicity, we have assumed that the island is deconstructed at rate r underwater, where r is the distance to the waterfront.

The height at the point of submersion is the same as the sea level, and so

$$f(a,t) = 0.01t.$$

Then, since $f(a,0) = 1 - a$, integrating over t gives,

$$0.01t = 1 - a(t) + \int_0^t a(t')\,\mathrm{d}t' - \int_0^t a(t)\,\mathrm{d}t'.$$

Observing that the final term is simply $ta(t)$, taking the derivative of both sides and rearranging gives

$$(1+t)\dot{a} = -0.01.$$

Since we now know the speed of the waterfront at all times, we can find out how long it takes for it to move 1 unit to the left by integrating under the speed–time curve, i.e. by solving

$$-1 = \int_0^t -\frac{0.01}{s+1}\,\mathrm{d}s,$$

which results in $100 = \ln(t+1) \implies t = e^{100} - 1$. $\square$

Problem 2.

(proposed by Simeon Kiflie)

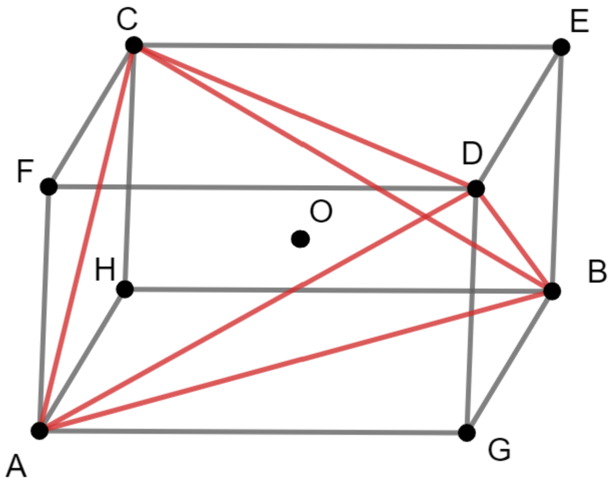
Show that if the distance between opposite edges of a tetrahedron is at least 1, then its volume is at least 1/3.

Notes on Marking. Most participants who drew the diagram in Solution 1 managed to complete the proof but some marks were awarded for anyone who drew the diagram at all. Several participants were able to obtain the expression of the volume of the tetrahedron as in solution 2 which were awarded partial marks. Only one participant was able to bound the cross product term and conclude via this method. No marks were awarded to anyone who only computed the volume of a regular tetrahedron satisfying the assumption.

Solution 1.

(solution by Dylan Toh)

Let A, B, C, D be the vertices of the tetrahedron. WLOG we may assume the centre of mass of the tetrahedron to be the origin O. Let E, F, G, H be the reflections of A, B, C, D about O respectively.



First, note that $AGBHFDEC$ is a parallelepiped. To see this, let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be the vectors to the vertices of the tetrahedron, with $\vec{a} + \vec{b} + \vec{c} + \vec{d} = 0$. Then

$$\overrightarrow{AF} = \overrightarrow{BE} = -\vec{b} - \vec{a} = \vec{c} + \vec{d} = \overrightarrow{HC} = \overrightarrow{GD}$$

and similarly for the other two sets of 4 edges of the parallelepiped.

Next, observe that the volume of the tetrahedron is 1/3 the volume of the parallelepiped. This is because a (volume-preserving) shear transformation followed by scaling sends the parallelepiped to the unit cube, of which the volume of the tetrahedron may then be computed to be 1/3 (e.g. by integrating to find the volume of an adjacent tetrahedron $GABD$ to be 1/6).

Finally, the distance between the lines through opposite edges of the tetrahedron is the distance between opposite faces of the parallelepiped. For instance, the planes $AFCH$ and $GDEB$ are spanned by $\overrightarrow{AC} = \overrightarrow{GE}$ and $\overrightarrow{FH} = \overrightarrow{DB}$.

The volume of the parallelepiped is

$$V = \text{Area}(AFCH) \cdot \text{distance}(AFCH, GDEB)$$

of which the latter distance between planes $AFCH$ and $GDEB$ is at least 1; and

$$\begin{aligned}\text{Area}(AFCH) &= |AF| \cdot (\text{distance between lines } CH \text{ and } AF) \\ &\geq \text{distance}(AHBG, FCED) \cdot \text{distance}(AFDG, HCEB)\end{aligned}$$

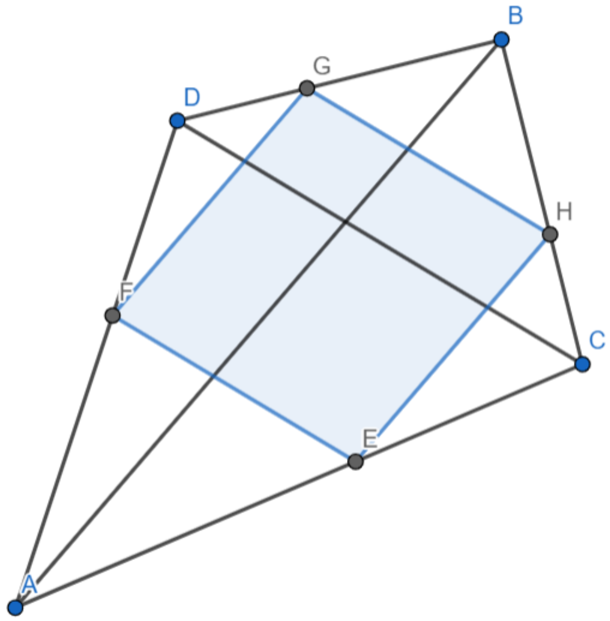
is at least 1 as well. Thus the volume of the parallelepiped is at least 1, and the volume of the tetrahedron $ABCD$ is at least 1/3. $\square$

Solution 2.

(solution by Matthew Johnson)

Let A, B, C, D be the vertices of the tetrahedron. WLOG we may rotate the tetrahedron so that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have no z-component. Additionally by translating we may assume WLOG that AB lies in the $z = 0$ plane and CD lies in the $z = h$ plane where $h \geq 1$.

Now consider the area of each z-slice of the tetrahedron. Let E, F, G, H respectively be the intersection of AC, AD, BD, BC with the $z = c \in [0, h]$ plane. It can be shown that $EFGH$ is a parallelogram with $FG = EH = \left(1 - \frac{c}{h}\right) AB$ and $EF = HG = \left(\frac{c}{h}\right) CD$, hence the area of the parallelogram at $z = c$ is $\left|\overrightarrow{FG} \times \overrightarrow{FE}\right| = \left(1 - \frac{c}{h}\right)\left(\frac{c}{h}\right)\left|\overrightarrow{AB} \times \overrightarrow{CD}\right|$. Integrating the area from $c = 0$ to $c = h$, we find that the volume of the tetrahedron is $\frac{h}{6}\left|\overrightarrow{AB} \times \overrightarrow{CD}\right| \geq \frac{1}{6}\left|\overrightarrow{AB} \times \overrightarrow{CD}\right|$.



To conclude it is enough to show that $\left|\overrightarrow{AB} \times \overrightarrow{CD}\right| \geq 2$. By intermediate value theorem, we can see that there exists $c' \in [0, h]$ such that $EFGH$ is a rhombus. Furthermore by the assumption we know $|EG|, |FH| \geq 1$ and we use the fact that $\left(1 - \frac{c}{h}\right)\left(\frac{c}{h}\right) \leq \frac{1}{4}$ for all $c \in [0, h]$. Putting it all together, we get $\frac{1}{2} \leq \frac{1}{2}|EG||FH| = \text{Area}(\text{Rhombus at } c') = \left(1 - \frac{c'}{h}\right)\left(\frac{c'}{h}\right)\left|\overrightarrow{AB} \times \overrightarrow{CD}\right| \leq \frac{1}{4}\left|\overrightarrow{AB} \times \overrightarrow{CD}\right|$ and we are done. $\square$

Problem 3.

(proposed by Ethan Tan)

The numbers $1, 2, \dots, n$ are written on a blackboard and then erased via the following process:

- Before any numbers are erased, a pair of numbers is chosen uniformly at random and circled.
- Each minute for the next $n-1$ minutes, a pair of numbers still on the blackboard is chosen uniformly at random and the smaller one is erased.
- In minute n, the last number is erased.

What is the probability that the smaller circled number is erased before the larger?

Notes on Marking. While only about half the contestants attempted the question, those who did fared well on average. Errors in otherwise complete solutions were rare. No marks were awarded for calculating the probability for specific n only, or for giving a (correct) Ansatz without justification.

Solution 1.

(solution by Ethan Tan)

Suppose that one of the circled numbers is 1. Let X_n be the minute in which 1 is erased (we count the first minute as minute 0); since one number is erased each minute and numbers $2, \dots, n-1$ are equally likely to be the other circled number x, we see that the probability 1 is erased before x is $1 - \mathbb{E}X_n/(n-1)$. Now $P(X_n = 0) = 2/n$, and otherwise we reach the same situation with $n-1$ numbers in total (note that in this case it doesn't matter which number was erased because all numbers beat 1). So we get

$$\mathbb{E}X_n = (1 - 2/n)(1 + \mathbb{E}X_{n-1}),$$

with $\mathbb{E}X_1 = \mathbb{E}X_2 = 0$. This solves to $\mathbb{E}X_n = (n-2)/3$. So the probability 1 is erased before x is $1 - (n-2)/3(n-1) = (2n-1)(3n-3)$.

In the general case, let p_n be the probability that the smaller number is erased first. 1 is circled with probability $2/n$. If 1 is not circled at the start, then we can effectively ignore it because any time it is used in a pair, it is erased. This reduces to the $n-1$ case. So

$$p_n = \frac{2}{n} \cdot \frac{2n-1}{3n-3} + \frac{n-2}{n} p_{n-1},$$

and $p_2 = 1$, which solves to $2(n+1)/3n$. □

Solution 2.

(solution by Gergely Rozgonyi)

We first check some simple cases. Let p_n be the probability for the case where we have n numbers on the blackboard. For $n = 2$, we have $p_2 = 1$. For $n = 3$, we have the following cases: if one of the circled numbers is 3, then we will certainly erase the smaller number before 3. If the circled numbers are 1 and 2, then the only way we can erase 2 before 1, is the case where in minute two the two chosen numbers are 2 and 3. Overall we have

$$p_3 = \frac{\binom{2}{1}}{\binom{3}{2}} + \frac{1}{\binom{3}{2}} \frac{1}{\binom{3}{2}} = \frac{7}{9}.$$

Now, suppose we have

$$p_k = \frac{2k+2}{3k} \qquad \forall\, 2 \le k \le n-1$$

and proceed by induction.

Let X_m, X_M denote the smaller and bigger number circled respectively, and T_m, T_M denote the minute they are erased. Then the following events form a partition:

$$\{T_m = 2\}; \qquad \{T_M = 2\}; \qquad \{T_m \neq 2, T_M \neq 2\};$$
$$P(T_m = 2) + P(T_M = 2) + P(T_m \neq 2, T_M \neq 2) = 1.$$

By the law of total probability, we can write p_n as

$$\begin{aligned}
p_n =& P(T_m < T_M) \\
=& P(T_m < T_M | T_m = 2)P(T_m = 2) + P(T_m < T_M | T_M = 2)P(T_M = 2) \\
&+ P(T_m < T_M | T_m \neq 2, T_M \neq 2)P(T_m \neq 2, T_M \neq 2) \\
=& 1 \cdot P(T_m = 2) + 0 \cdot P(T_M = 2) + p_{n-1} \cdot P(T_m \neq 2, T_M \neq 2).
\end{aligned}$$

The first and second terms are clear, and for the third one, we note that choosing two numbers from $1, \ldots, n-1$ (as we fixed that we erase a number that was not circled) uniformly at random is independent of the erasure order, and hence after erasing the first number, the game plays as one with $n-1$ numbers written on the blackboard. We now need to calculate the probabilities for the partition. Note that if $X_m = k$, we erase X_m at $t = 2$ if we pick X_m and a number that is greater than X_m. This happens with probability

$$\frac{n-k}{\binom{n}{2}}.$$

The same result holds for $X_M = k$. Using the law of total probability, we have

$$\begin{aligned}
P(T_m = 2) &= \sum_{1 \le k < K \le n} P(T_m = 2 | X_m = k, X_M = K) P(X_m = k, X_M = K) \\
&= \sum_{1 \le k < K \le n} \frac{n-k}{\binom{n}{2}} \frac{1}{\binom{n}{2}} = \frac{1}{\binom{n}{2}^2} \sum_{k=1}^{n-1} \sum_{K=k+1}^{n} (n-k) \\
&= \frac{1}{\binom{n}{2}^2} \sum_{k=1}^{n-1} (n-k)^2 = \frac{1}{\binom{n}{2}^2} \sum_{k=1}^{n-1} k^2 = \frac{1}{\binom{n}{2}^2} \frac{(n-1)n(2n-1)}{6} = \frac{2n-1}{3\binom{n}{2}}; \\
P(T_M = 2) &= \sum_{1 \le k < K \le n} P(T_M = 2 | X_m = k, X_M = K) P(X_m = k, X_M = K) \\
&= \sum_{1 \le k < K \le n} \frac{n-K}{\binom{n}{2}} \frac{1}{\binom{n}{2}} = \frac{1}{\binom{n}{2}^2} \sum_{K=2}^{n} \sum_{k=1}^{K-1} (n-K) \\
&= \frac{1}{\binom{n}{2}^2} \sum_{K=2}^{n} (K-1)(n-K) = \frac{1}{\binom{n}{2}^2} \sum_{K=2}^{n} \left[-(K-1)^2 + (K-1)(n-1) \right] \\
&= \frac{1}{\binom{n}{2}^2} \frac{n(n-1)(n-2)}{6} = \frac{n-2}{3\binom{n}{2}};
\end{aligned}$$

$$P(T_m \neq 2, T_M \neq 2) = 1 - P(T_m = 2) - P(T_M = 2) = \cdots = \frac{n-2}{n}.$$

Using these results and the induction hypothesis, we have

$$p_n = \frac{2(2n-1)}{3n(n-1)} + \frac{n-2}{n} \frac{2n}{3n-3} = \frac{2n+2}{3n},$$

and the result follows. $\square$

Alternative branching solution: Instead of calculating the probability for small n and then conjecturing the correct form of the solution to use for induction, one could note just that the case $n = 2$ is trivial, and calculate p_n as a finite sum plus a telescoping product iterating from p_2.

Solution 3.

(solution by Dylan Toh)

Let $(X_1, \ldots, X_n)$ be the numbers listed in erasure order. As the circling of the numbers is independent of the erasure process, the pair of circled numbers may be taken to be chosen uniformly from $X_1, \ldots, X_n$. Consequently, the probability that the smaller circled number was erased before the larger is

$$\begin{aligned} p_n &= \sum_{1 \le i < j \le n} P(X_i, X_j \text{ are circled}) \cdot P(X_i < X_j) \\ &= \binom{n}{2}^{-1} \mathbb{E}\left[\sum_{1 \le i < j \le n} \mathbb{1}_{X_i < X_j}\right] = \binom{n}{2}^{-1} \mathbb{E}\left[A\right], \end{aligned}$$

where $A = \#\{(i,j) : 1 \le i < j \le n, X_i < X_j\}$ is the number of ascending pairs of X. Note

$$A = \sum_{i=1}^{n-1} \#\{j : i < j \le n, X_i < X_j\} = \sum_{i=1}^{n-1} (m_i - 1),$$

where X_i is the M_i-th largest number among the $(n - i + 1)$ numbers on the board at the i-th minute. Consequently,

$$p_n = \binom{n}{2}^{-1} \sum_{i=1}^{n-1} \mathbb{E}\left[M_i - 1\right].$$

We first compute the distribution of X_1. Note that for $m = 1, \ldots, n$,

$$P(X_1 \ge n + 1 - m) = P(\text{circled numbers are both} \ge n + 1 - m) = \binom{n}{2}^{-1} \binom{m}{2},$$

so by taking adjacent differences,

$$P(X_1 = n + 1 - m) = \binom{n}{2}^{-1} \binom{m}{2} - \binom{n}{2}^{-1} \binom{m-1}{2} = \binom{n}{2}^{-1} (m - 1)$$

for $m = 1, 2, \ldots, n$. More generally, for each $1 \le i \le n$, M_i is distributed among $(1, 2, \ldots, n - i + 1)$ with probabilities $\binom{n-i+1}{2}^{-1}(0, 1, \ldots, n - i)$. Its expectation may thus be computed:

$$\begin{aligned} \mathbb{E}[M_i - 1] &= \sum_{m=1}^{n-i+1} \binom{n-i+1}{2}^{-1} (m-1) \cdot (m-1) \\ &= \frac{2}{(n-i+1)(n-i)} \cdot \frac{(n-i)(n-i+1)(2(n-i)+1)}{6} = \frac{2(n-i)+1}{3}. \end{aligned}$$

Finally, the desired probability is

$$p_n = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \frac{2(n-i)+1}{3} = \frac{2}{3n(n-1)} \cdot (n(n-1) + (n-1)) = \frac{2(n+1)}{3n}. \quad \square$$

Problem 4.

(proposed by Dylan Toh)

Do there exist infinitely many positive integers m such that the sum of the positive divisors of m (including m itself) is a perfect square?

Notes on Marking. One mark was awarded for stating that $\sigma(ab) = \sigma(a)\sigma(b)$ or equivalent, such as the formula based on prime factorisations (here, σ is the sum-of-divisors function). Unfortunately, no marks were awarded for solutions assuming an infinite number of Mersenne primes, primes generated by a quadratic equation, or prime pairs related by a linear relation.

Solution 1.

(solution by Tejas Mittal and Ahmed Ittihad Hasib [Oxford])

We show that the answer is yes. Suppose for the sake of contradiction that there are finitely many such positive integers m, with the largest being M. (Note that this set is non-empty since 22 works.) We will show how to construct a larger number whose sum of divisors is a square.

Let σ be the sum-of-divisors function, and note that $\sigma(ab) = \sigma(a)\sigma(b)$ whenever a and b are coprime. Let p_i denote the i-th prime. For each $i \in \{0, 1, 2, \ldots\}$, let a_i be the smallest positive integer such that $p_i^{a_i} > M$. Let k be such that p_k is the largest prime less than or equal to M, and let n be such that $p_n > \max\{\sigma(p_1^{a_1}), \sigma(p_2^{a_2}), \ldots, \sigma(p_k^{a_k})\}$.

Now since $\sigma(p_i^{a_i}) \le p_n + 1$ for all $i \in \{1, 2, \ldots, n\}$, $\sigma(p_i^{a_i})$ is too small to have any prime divisors greater than p_{n-1}, so we can write each $\sigma(p_i^{a_i})$ as $p_1^{b_{i,1}} p_2^{b_{i,2}} \cdots p_{n-1}^{b_{i,n-1}}$. Hence, for any $I \subseteq \{1, 2, \ldots, n\}$, we have that

$$\sigma\left(\prod_{i\in I} p_i^{a_i}\right) = \prod_{i\in I} \sigma(p_i^{a_i}) = p_1^{c_1} \cdots p_{n-1}^{c_{n-1}}, \qquad \text{where } c_j = \sum_{i\in I} b_{i,j},$$

so it suffices to show that there exists a non-empty $I \subseteq \{1, 2, \ldots, n\}$ such that all c_j are even.

Let S_n denote the subsets of $\{1, 2, \ldots, n\}$, and consider the function

$$\begin{aligned} f : S_n &\to \{0, 1\}^{n-1} \\ I &\mapsto (c_1 \bmod 2, \ldots, c_{n-1} \bmod 2). \end{aligned}$$

We wish to find an $I \in S_n$ such that $f(I) = \mathbf{0}$. Since the domain has cardinality 2^n while the codomain has cardinality 2^{n-1}, there exist distinct $J, K \in S_n$ such that $f(J) = f(K)$ by pigeonhole principle. Let $J' = J\backslash(J \cap K)$ and $K' = K\backslash(J \cap K)$, and note that we still have $f(J') = f(K')$, but now J' and K' are distinct and disjoint. Hence, $f(J' \cup K') = \mathbf{0}$, and so finally letting $I = J' \cup K'$ we have

$$\sigma\left(\prod_{i\in I} p_i^{a_i}\right) = \sigma\left(\prod_{j\in J'} p_j^{a_j}\right)\sigma\left(\prod_{k\in K'} p_k^{a_k}\right) = p_1^{c_1} \cdots p_{n-1}^{c_{n-1}}, \qquad \text{where all } c_i \text{ are even.}$$

Since each $p_i^{a_i} > M$, it follows that $\prod\limits_{i\in J'\cup K'} p_i^{a_i}$ is also greater than M, and so we are done. $\square$

Solution 2.

(solution by Dylan Toh and Tony Wang)

We show that the answer is yes. Let $\sigma(n)$ be the sum of divisors of n, and note that if $p_1, \ldots, p_k$ are k distinct primes, then $\sigma(p_1 \ldots p_k) = (p_1 + 1) \ldots (p_k + 1)$.

Let p_i denote the i-th prime, and let P_N be the set of all primes less than N. For an odd prime p, $p+1$ is even, and so $p \in P_N$ implies that all prime factors of p are in $P_{(N+1)/2}$.

For a fixed large N, let $n = |P_N|$ and $m = \left|P_{(N+1)/2}\right|$. Consider the linear map over the field with two elements $\Phi : \mathbb{F}_2^n \to \mathbb{F}_2^m$ defined by the matrix M over $\mathbb{F}_2$ where

$$M_{i,j} = \begin{cases} 0 & \text{if the exponent of } p_j \text{ in the prime factorisation of } p_i + 1 \text{ is even, and} \\ 1 & \text{if the exponent of } p_j \text{ in the prime factorisation of } p_i + 1 \text{ is odd.} \end{cases}$$

Identifying $\mathbb{F}_2$ with $\{0, 1\}$, note that for all $\alpha \in \{0, 1\}^n$,

$$\sigma \left(\prod_{i=1}^{n} q_i^{\alpha_i} \right) = A^2 \prod_{j=1}^{m} q_j^{(\Phi(\alpha))_j}, \quad \text{for some } A \in \mathbb{N}.$$

Consequently, each α in the kernel of Φ corresponds to a unique square-free integer $\prod_{i=1}^{n} q_i^{\alpha_i}$ whose sum of divisors is a perfect square.

Finally, since $n - m = \left|\{p : p \text{ prime}, \frac{N+1}{2} < p \leq N\}\right| \to \infty$ as $N \to \infty$ by the prime number theorem, we have $|\ker \Phi| \geq 2^{n-m} \to \infty$ as $N \to \infty$ as well. Therefore, there are infinitely many integers whose sum of divisors is a perfect square. $\square$

Problem 5.

(proposed by Dylan Toh)

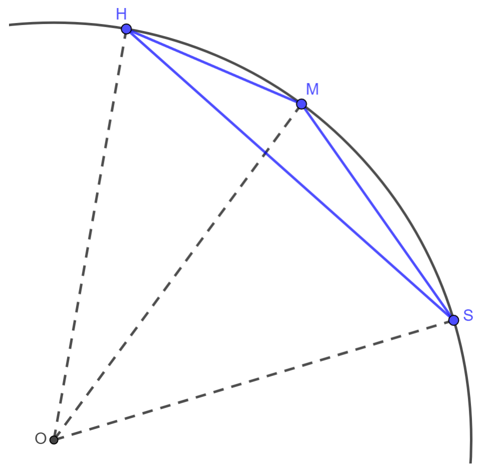
A clock has an hour, minute, and second hand, all of length 1. Let T be the triangle formed by the ends of these hands. A time of day is chosen uniformly at random. What is the expected value of the area of T?

Notes on Marking. The correct numerical answer was worth 1 mark. Parameterising of the continuous hand movements with a single variable (e.g. time) and setting up an integral expression for the expected area of the determined triangle was worth 1 mark. Evaluating the integral on segments on which the signed area is of fixed sign was worth 1 mark. Determining the exact times at which the area flips sign (or exploiting related symmetries) was worth 1 mark. Establishing the Lemma below regarding interleaving of sign-flip times was worth 3 marks.

Solution 1.

(solution by Dylan Toh)

WLOG scale the time of the day such that 12 hours pass in time 2π. The hour, minute, and second hands then move clockwise at angular speeds $1, 12$, and 720 respectively.



We first derive two expressions for the area of the desired triangle. Let the centre of the clock be O, and the tips of the hour, minute, and second hands be H, M, S respectively. At time $t \in [0, 2\pi]$, one has

$$\angle MOS = 708t, \quad \angle HOM = 11t, \quad \angle HOS = 719t$$

where angles are directed clockwise. Thus

$$[SMH] = [MOS] + [HOM] - [HOS] = \frac{1}{2}\left(\sin 708t + \sin 11t - \sin 719t\right)$$

is the signed area of the triangle.

Alternatively, noting that $\angle HSM = \frac{1}{2}\angle HOM = \frac{11t}{2} \pmod{\pi}$, the absolute area of the triangle may be computed by sine rule:

$$|[SMH]| = \frac{1}{2} \cdot |SH| \cdot |SM| \cdot |\sin \angle HSM| = \left|\frac{1}{2} \cdot 2\sin\frac{719t}{2} \cdot 2\sin\frac{708t}{2} \cdot \sin\frac{11t}{2}\right|.$$

Thus the signed area is

$$[SMH] = 2\sin\frac{11t}{2}\sin\frac{708t}{2}\sin\frac{719t}{2}$$

which may alternatively be shown directly by trigonometric identities on the first expression. Crucially, observe that the sign ($\pm$) of $[SMH]$ flips at integer multiples of $\frac{2\pi}{11}$, $\frac{2\pi}{708}$, and $\frac{2\pi}{719}$.

Let $(a,b,c) = (11,708,719)$; more generally, we consider coprime positive integers a,b,c with $a+b=c$. We wish to compute the average area of the triangle:

$$\langle A\rangle = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}(\sin at + \sin bt - \sin ct)\cdot s(t)dt$$

where $s(t) \in \{\pm 1\}$ is positive at $t = 0^+$, and swaps sign at integer multiples of $\frac{2\pi}{a}$, $\frac{2\pi}{b}$, and $\frac{2\pi}{c}$.

Lemma. If $0 = t_0 < t_1 < t_2 < \cdots < t_{a+b-2} < t_{a+b-1} = 2\pi$ are the integer multiples of $\frac{2\pi}{a}$, $\frac{2\pi}{b}$ in $[0,2\pi]$ rearranged in ascending order, then $t_{k-1} < \frac{2\pi k}{c} < t_k$ for all $k = 1,2,\ldots,a+b-1$; i.e.

$$0 < \frac{2\pi}{c} < t_1 < \frac{4\pi}{c} < t_2 < \frac{6\pi}{c} < \cdots < t_{c-2} < \frac{2\pi(c-1)}{c} < 1.$$

Proof of Lemma. Since a,b,c are coprime, thus the values $\{\frac{i}{a} : i = 1,\ldots,a-1\}$, $\{\frac{j}{b} : j = 1,\ldots,b-1\}$, and $\{\frac{k}{c} : k = 1,\ldots,c-1\}$ are distinct in $(0,1)$. It suffices to show that there is at least one multiple $\frac{2\pi l}{c}$ between each t_{k-1} and t_k, for each $k = 1,2,\ldots,c-1$; if this is true, then since there are only $c-1$ such multiples, they must fall exactly in place ($l = k$). This may be justified with a brief case check:

- If t_{k-1} and t_k both have denominator a, then the interval $[t_{k-1}, t_k]$ has length $\frac{2\pi}{a} \geq \frac{2\pi}{c}$, and thus must contain an integer multiple of $\frac{2\pi}{c}$.
- Similarly, if t_{k-1} and t_k both have denominator b, then the interval $[t_{k-1}, t_k]$ has length $\frac{2\pi}{b} \geq \frac{2\pi}{c}$, thus contains an integer multiple of $\frac{2\pi}{c}$.
- Finally, if t_{k-1} and t_k are $\frac{2\pi i}{a}$ and $\frac{2\pi j}{b}$ in some order, then since $\frac{i+j}{a+b} = \frac{i+j}{c}$ lies between $\frac{i}{a}$ and $\frac{j}{b}$, thus $[t_{k-1}, t_k]$ contains the integer multiple $\frac{2\pi(i+j)}{c}$.

The lemma thus allows us to evaluate the integral in pieces:

$$\begin{aligned}
\langle A\rangle &= \frac{1}{4\pi}\sum_{k=1}^{c-1}\left(\int_{t_{k-1}}^{\frac{2\pi k}{c}} dt - \int_{\frac{2\pi k}{c}}^{t_k} dt\right)(\sin at + \sin bt - \sin ct)\\
&= \frac{1}{4\pi}\sum_{k=1}^{c-1}\left(\left[-\frac{\cos at}{a} - \frac{\cos bt}{b} + \frac{\cos ct}{c}\right]_{t_{k-1}}^{\frac{2\pi k}{c}} - \left[-\frac{\cos at}{a} - \frac{\cos bt}{b} + \frac{\cos ct}{c}\right]_{\frac{2\pi k}{c}}^{t_k}\right)\\
&= \frac{1}{4\pi a}\sum_{k=1}^{c-1}\left(\cos at_{k-1} - 2\cos\frac{2\pi ak}{c} + \cos at_k\right) + \ldots \text{(similar expressions for } b \text{ and } c\text{)}.
\end{aligned}$$

We evaluate the trigonometric sums using the identity

$$\Lambda_{m,n} := \sum_{x=0}^{n-1}\cos\frac{2\pi mx}{n} = \begin{cases} n, & n \mid m \\ 0, & n \nmid m \end{cases}$$

which follows from considering the real part of the geometric progression $e^{2\pi imx/n}$.

Since $\{t_1,\ldots,t_{c-2}\} = \{\frac{2\pi i}{a} : i = 1,\ldots,a-1\} \cup \{\frac{2\pi b}{a} : j = 1,\ldots,b-1\}$, we may carefully collect terms to obtain

$$\begin{aligned}
\langle A\rangle &= \frac{1}{4\pi}\left(\frac{2\Lambda_{a,a} + 2\Lambda_{a,b} - 2\Lambda_{a,c}}{a} + \frac{2\Lambda_{b,a} + 2\Lambda_{b,b} - 2\Lambda_{b,c}}{b} - \frac{2\Lambda_{c,a} + 2\Lambda_{c,b} - 2\Lambda_{c,c}}{c}\right)\\
&= \frac{1}{2\pi}\left(\frac{a}{a} + \frac{b}{b} + \frac{c}{c}\right) = \frac{3}{2\pi}. \qquad \square
\end{aligned}$$

Comments.

(i) The lemma regarding interlacing of the values at which the area swaps sign is an instance of the Rayleigh-Beatty theorem.

(ii) The final answer nicely agrees with the expected area of the triangle formed by 3 points uniformly chosen along the circumference of the unit circle.

(iii) If $a = 1$, the final calculation includes extra non-zero terms ($\Lambda_{b,1}$ and $\Lambda_{c,1}$), resulting in a slightly larger expected area $\langle A \rangle = \frac{3}{2\pi}\left(1 + \frac{1}{b(b+1)}\right)$.

Solution 2.

(solution by Ethan Tan, sketch only)

As before, we have

$$\langle A \rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} (\sin at + \sin bt - \sin ct) \cdot s(t) dt.$$

Note that $s(t) = f(at)f(bt)f(ct)$, where $f(t)$ os a square wave with period 2π and amplitude 1. By Fourier decomposition, we can write

$$f(t) = \frac{2}{\pi} \sum_{n \in \mathbb{Z},\ n \text{ odd}} e^{int}/n.$$

Hence we have

$$s(t) = f(at)f(bt)f(ct) = \frac{8}{\pi^3 abc} \sum_k e^{ikt} \sum_{\substack{an+bm+cl=k \\ n,m,l \in 2\mathbb{Z}+1}} \frac{1}{nml}.$$

Since $\sin at = (e^{iat} - e^{-iat})/2$, and similarly for $\sin bt$, $\sin ct$, it suffices to extract the coeffieicnt of e^{iat} (noting that the coefficient of e^{-iat} is the negative of the coefficient of e^{iat} since s is odd). One can fix a value of n, compute the sum over pairs (m, l) such that $an + bm + cl = k$ with m, l odd (these points lie on a line and so we need to compute the sum of the inverses of a quadratic in m which can be done without too much trouble), and then sum back over n. □