IMPERIAL-CAMBRIDGE
MATHEMATICS
COMPETITION

ROUND ONE

26 November 2023

Name: ________________________________

Contestant ID: ________________________________

University: ________________________________

Instructions:

- Do not turn over until told to do so.
- You will have 3 hours to solve 6 problems, each of which carries 10 marks.
- Use a black or blue pen or a dark pencil. Rulers, compasses, protractors, and erasers may be used but will not be required. All electronic devices, including calculators, are prohibited.
- Drinks are allowed, but food is prohibited.
- Write your solution to each problem on a different page. At the top of each page, write down the question number, your initials, and your contestant number. Use both sides whenever possible. Writing clearly and not too faintly – your work will be scanned for marking.
- Problems are listed roughly in order of difficulty. Proofs are expected for all problems even if they only ask for an answer.
- One complete solution will be awarded more marks than several unfinished attempts.
- You may not leave the contest venue early unless exceptional circumstances arise.
- Do not take away the problems sheet or any rough work when leaving the venue.

Declaration:

I hereby declare that the work I am submitting is entirely my own, and that I will not discuss or disclose the contents of this paper until 14:00, 26 November 2023, GMT Time.

Signature: ________________________________

Problem 1. Define the Fibonacci numbers recursively by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that 3^{2023} divides

$$3^2 \cdot F_4 \;+\; 3^3 \cdot F_6 \;+\; 3^4 \cdot F_8 \;+\; \cdots \;+\; 3^{2023} \cdot F_{4046}.$$

Problem 2. Fredy starts at the origin of the Euclidean plane. Each minute, Fredy may jump a positive integer distance to another lattice point, provided the jump is not parallel to either axis. Can Fredy reach any given lattice point in 2023 jumps or less?

Note: The x- and y-axes of the Euclidean plane are fixed. A lattice point is a point (m, n) with integer coordinates $m, n \in \mathbb{Z}$.

Problem 3. There are 10^5 users on the social media platform Mathsenger, every pair of which has a direct messaging channel. Prove that each messaging channel may be assigned one of 100 encryption keys, such that no 4 users have the 6 pairwise channels between them all being assigned the same encryption key.

Note: Partial marks will be awarded if the result is proved with the value 100 replaced with 1000 or 10000.

Problem 4. Points A, B, C, and D lie on the surface of a sphere with diameter 1. What is the maximum possible volume of tetrahedron $ABCD$?

Problem 5.

(a) Is there a non-linear integer-coefficient polynomial $P(x)$ and an integer N such that all integers greater than N may be written as the greatest common divisor of $P(a)$ and $P(b)$ for positive integers a and b with $a > b$?

(b) Is there a non-linear integer-coefficient polynomial $Q(x)$ and an integer M such that all integers greater than M may be written as $Q(a) - Q(b)$ for positive integers a and b with $a > b$?

Problem 6. Let $f : \mathbb{N} \to \mathbb{N}$ be a bijection of the positive integers. Prove that, as $N \to +\infty$, at least one of the limits

$$\sum_{n=1}^{N} \frac{1}{n + f(n)} \to +\infty \qquad \text{or} \qquad \sum_{n=1}^{N} \frac{f(n) - n}{n f(n)} \to +\infty$$

is true.

Note: The function $f : \mathbb{N} \to \mathbb{N}$ is a bijection if, for every positive integer a, there is a unique positive integer n such that $f(n) = a$.