IMPERIAL-CAMBRIDGE

MATHEMATICS

COMPETITION

7th Edition (2023–2024)

ROUND ONE

Official Solutions*

Last updated: 27 Dec 2023, 4pm

*A solution may receive full or partial marks even if it does not appear in this booklet.

Problem 1. *(proposed by Dylan Toh)*

Define the Fibonacci numbers recursively by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that 3^{2023} divides

$$3^2 \cdot F_4 \ + \ 3^3 \cdot F_6 \ + \ 3^4 \cdot F_8 \ + \ \cdots \ + \ 3^{2023} \cdot F_{4046}.$$

Notes on Marking. Few partial marks were awarded for this problem. Quoting Binet's formula was not worth any any marks by itself, but 2 marks were awarded for using it to sum the geometric series. Several useful facts about the expression in question were awarded 1 mark when stated. Although induction was useful in many solutions, an attempt to induct that did not produce any useful results was not awarded any marks. No marks were awarded for determining the Fibonacci numbers modulo three. Calculational errors were penalised when deemed major enough.

Solution 1. *(solution by Dylan Toh of Cambridge)*

We induct on $n \geq 2$ that

$$S(n) := 3^2 \cdot F_4 \ + \ 3^3 \cdot F_6 \ + \ 3^4 \cdot F_8 \ + \ \cdots \ + \ 3^n \cdot F_{2n} = 3^{n+1} \cdot F_{2n-2}.$$

- **Base case:** For $n = 2$, it is indeed true that $3^2 \cdot F_4 = 3^3 \cdot F_2$.
- **Inductive Step:** Since $S(n+1) = S(n) + 3^{n+1} \cdot F_{2n+2}$, it suffices to show that

  $$3^{n+1} \cdot F_{2n-2} + 3^{n+1} \cdot F_{2n+2} = 3^{n+2} \cdot F_{2n}.$$

  Equivalently, dividing by 3^{n+1}, it suffices to show for all $n \geq 2$, $F_{2n-2} + F_{2n+2} = 3 \cdot F_{2n}$. This follows directly from definitions:

  $$\begin{aligned} F_{2n-2} + F_{2n+2} &= (F_{2n} - F_{2n-1}) + (F_{2n} + F_{2n+1}) \\ &= 2 \cdot F_{2n} + (F_{2n+1} - F_{2n-1}) \\ &= 3 \cdot F_{2n}. \end{aligned}$$

Now, the result for $n = 2023$ directly implies that 3^{2023} divides $S(2023)$, as desired □

Solution 2. *(solution by Dylan Toh)*

The Fibonacci numbers are coefficients of the formal power series

$$f(x) = \frac{1}{1 - x - x^2} = \sum_{n \geq 0} F_{n+1} x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \cdots$$

Substituting $x = \pm\sqrt{3}$, one obtains the identity

$$\begin{aligned} f(\sqrt{3}) - f(-\sqrt{3}) &= \frac{1}{-2 - \sqrt{3}} - \frac{1}{-2 + \sqrt{3}} = 2\sqrt{3} \\ &= 2\sqrt{3} \sum_{n \geq 0} F_{n+1} \left((\sqrt{3})^n - (-\sqrt{3})^n \right) = \frac{2}{\sqrt{3}} \sum_{n \geq 1} F_{2n} 3^n \end{aligned}$$

where the series converges in the $(\sqrt{3})$-adic numbers. Rearranging,

$$\sum_{n\geq 2} F_{2n}3^n = -3 + \sum_{n\geq 1} F_{2n}3^n = 0.$$

This equality then makes sense in the 3-adic numbers. Since $\nu_3(F_{2n}3^n) \geq 2024$ for $n \geq 2024$, thus

$$\nu_3\left(\sum_{n=2}^{2023} F_{2n}3^n\right) = \nu_3\left(-\sum_{n\geq 2024} F_{2n}3^n\right) \geq 2024 \geq 2023$$

which is the desired result. □

Solution 3. *(solution by contestants)*

Let

$$a = 3^2F_4 + 3^3F_6 + 3^4F_8 + \cdots + 3^{2023}F_{4046}.$$

Then note that

$$\begin{aligned}
2a &= 3\cdot a - a\\
&= 3^2(-F_4) - 3^3(F_6 - F_4) - \cdots - 3^{2023}(F_{4046} - F_{4044}) + 3^{2024}F_{4046}\\
&= -27 - 3^3F_5 - 3^4F_7 - \cdots - 3^{2023}F_{4045} + 3^{2024}F_{4046}.
\end{aligned}$$

Repeating the procedure,

$$\begin{aligned}
4a &= 3\cdot 2a - 2a\\
&= -54 + 3^3F_5 + 3^4(F_7 - F_5) + \cdots + 3^{2023}(F_{4045} - F_{4043}) + 3^{2024}(-F_{4046} - F_{4045})\\
&\quad + 3^{2025}F_{4046}\\
&= 81 + 3^4F_6 + \cdots + 3^{2023}F_{4044} + 3^{2024}(-F_{4047}) + 3^{2025}F_{4046}\\
&= 3a + 3^{2024}(-F_{4047} + 2F_{4046})
\end{aligned}$$

which implies that a is a multiple of 3^{2023}. □

Solution 4. *(solution by contestants)*

The closed form expression for the Fibonacci numbers is

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}},$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = -\varphi^{-1} = \frac{1-\sqrt{5}}{2}$ are the roots of the quadratic $x^2 + x - 1 = 0$. Thus

$$\begin{aligned}
\sum_{n=2}^{2023} 3^nF_{2n} &= \frac{1}{\sqrt{5}}\left[\sum_{n=2}^{2023} 3^n\varphi^{2n} - \sum_{n=2}^{2023} 3^n\psi^{2n}\right]\\
&= \frac{3^2}{\sqrt{5}}\left[\varphi^4\frac{(3\varphi^2)^{2022}-1}{3\varphi^2-1} - \psi^4\frac{(3\psi^2)^{2022}-1}{3\psi^2-1}\right]\\
&= \frac{3^2}{\sqrt{5}}\left[(3\varphi^2)^{2022} - (3\psi^2)^{2022}\right]\\
&= \frac{3^{2024}}{\sqrt{5}}\left[\varphi^{4044} - \psi^{4044}\right]\\
&= 3^{2024}\cdot F_{4044}
\end{aligned}$$

which implies the desired divisibility result. The key intermediate step arises from the identity

$$\frac{\varphi^4}{3\varphi^2-1} = \frac{\psi^4}{3\psi^2-1} = 1,$$

which is true by direct computation, or alternatively follows from the polynomial factorisation $x^4 - 3x^2 + 1 = (x^2 + x - 1)(x^2 - x - 1)$ which shows that φ and ψ are roots of the quartic, as desired. □

Solution 5.

(solution by contestants)

Let $S_1 = \sum_{i=2}^{2023} 3^i F_{2i}$ and $S_2 = \sum_{i=2}^{2023} 3^i F_{2i+1}$. Then,

$$3(S_1 + S_2) = \sum_{i=2}^{2023} 3^{i+1}(F_{2i} + F_{2i+1}) = \sum_{i=2}^{2023} 3^{i+1} F_{2i+2} = 3^{2024}F_{4048} + S_1 - 3^2 F_4. \tag{1}$$

Also note that

$$S_2 = \sum_{i=2}^{2023} 3^i (F_{2i} + F_{2i-1}) = S_1 + 3\left(\sum_{i=1}^{2022} 3^i F_{2i+1}\right) = S_1 + 3\left(S_2 - 3^{2023}F_{4047} + 3F_3\right). \tag{2}$$

Solving equations (1) and (2) yields $S_1 = 3^{2024}(2F_{4048} - 3F_{4047})$ and we are done. □

Solution 6.

(solution by Tudor-Ioan Caba of Oxford)

Let F be the Fibonacci matrix defined by

$$F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, F_n is the off-diagonal entry of F^n and so our sum S is the off-diagonal entry of

$$3^2 \cdot F^4 \;+\; 3^3 \cdot F^6 \;+\; 3^4 \cdot F^8 \;+\; \cdots \;+\; 3^{2023} \cdot F^{4046} = 3^2 \cdot F^4 \left((3F^2)^{2022} - I \right) (3F^2 - I)^{-1}.$$

As $F^4 = 3F^2 - I$, S is the off-diagonal entry of $3^{2024}F^{4044} - 9I$ which is the same as the off-diagonal entry of $3^{2024}F^{4044}$, which is $3^{2024}F_{4044}$. Clearly, this is a multiple of 3^{2023}. □

Solution 7.

(solution by Pietro Gualdi of SNS, Pisa, and Ahmed Ittihad Hasib)

By Binet's formula, $F_n = \dfrac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}$. Then, letting $S_n := \sum_{k=2}^{n} 3^k F_{2k}$, we have that

$$\begin{aligned}
(S_n + 3)\sqrt{5} &= \sum_{k=0}^{n} 3^k \left(\phi^{2k} - (-\phi)^{-2k}\right) \\
&= \sum_{k=0}^{n} \left(\left(\frac{9+3\sqrt{5}}{2}\right)^k - \left(\frac{9-3\sqrt{5}}{2}\right)^k \right) \\
&= \frac{\left(\frac{9+3\sqrt{5}}{2}\right)^{n+1} - 1}{\frac{9+3\sqrt{5}}{2} - 1} - \frac{\left(\frac{9-3\sqrt{5}}{2}\right)^{n+1} - 1}{\frac{9-3\sqrt{5}}{2} - 1}.
\end{aligned}$$

Multiplying both sides by $\left(\frac{7+3\sqrt{5}}{2}\right)\left(\frac{7-3\sqrt{5}}{2}\right)=1$, we get that

$$(S_n+3)\sqrt{5}=\left(\left(\frac{9+3\sqrt{5}}{2}\right)^{n+1}-1\right)\left(\frac{7-3\sqrt{5}}{2}\right)-\left(\left(\frac{9-3\sqrt{5}}{2}\right)^{n+1}-1\right)\left(\frac{7+3\sqrt{5}}{2}\right).$$

Now we work in the ring of integers of $\mathbb{Q}[\sqrt{5}]$, which is $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$. Consider the ideal $I=\langle 3^{n+1}\rangle$. Notice that $\left(\frac{9+3\sqrt{5}}{2}\right)^{n+1}, \left(\frac{9-3\sqrt{5}}{2}\right)^{n+1} \in I$. So, we have that

$$(S_n+3)\sqrt{5}\equiv\left(\frac{3\sqrt{5}-7}{2}\right)+\left(\frac{3\sqrt{5}+7}{2}\right)\equiv 3\sqrt{5} \mod I$$

Hence, $S_n\sqrt{5}\in I$. So, $S_n\sqrt{5}=3^{n+1}u$ for some $u\in\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$. Looking at norms, we must have $u=u'\sqrt{5}$. Which implies $S_n\in I$. So, $S_n=3^{n+1}\left(\frac{a+b\sqrt{5}}{2}\right)$ for $a,b\in\mathbb{Z}$ with same parity. It is easy to see that $b=0$ and hence, $3^{n+1}|S_n$ in $\mathbb{Z}$. $\square$

Problem 2. *(proposed by Tony Wang)*

Fredy starts at the origin of the Euclidean plane. Each minute, Fredy may jump a positive integer distance to another lattice point, provided the jump is not parallel to either axis. Can Fredy reach any given lattice point in 2023 jumps or less?

Note: The x- and y-axes of the Euclidean plane are fixed. A lattice point is a point (m, n) with integer coordinates $m, n \in \mathbb{Z}$.

Notes on Marking. No marks were awarded for stating that Fredy can reach any lattice point. 2 marks were awarded for noticing that small Pythagorean triangles become useful, either in solving the modulo problem below, or for constructing elementary jumps like $(0, 1)$ or $(1, 1)$. 3 marks were awarded for stating/using that Pythagorean triples can be scaled by any (integer) number. 3 marks were awarded for reducing the problem to a modulo-solvable finite grid. 2 marks were finally awarded for fully solving the finite grid problem. Note that those who constructed an axial jump and stated scaling implicitly solved such a modulo problem and hence got full marks. Marks were awarded for any of the previously mentioned steps, even if the final conclusion of the proof was incorrect (that Fredy couldn't reach the destination in time).

Solution 1. *(solution by Fredy Yip)*

For any integer $n \neq -1, 0, 1$, we may jump by $(n^2 - 1, 2n)$ and then $(-(n^2 - 1), 2n)$ to achieve a total displacement of $(0, 4n)$ in 2 moves. We may similarly do this vertically and hence achieve a displacement of $(4m, 4n)$ for any integers $n, m \neq -1, 0, 1$ in 4 moves.

Composing 2 such sequences of moves gives rise to a displacement of $(4m, 4n)$ for any integers n, m in 8 moves.

Working mod 4 in both coordinates, a move of $(-3, 4)$ is equivalent to moving by $(1, 0)$ and similarly we can move by $(\pm 1, 0)$, $(0, \pm 1)$ mod 4. Hence we may reach a point congruent to the destination point modulo 4 in four moves or less, after which we may reach the destination point in 8 moves. Hence a total of at most 12 moves suffices. $\square$

Solution 2. *(solution by Gergely Rozgonyi)*

For any integer $n \neq 0$, we consider the following sequence of jumps: $(4n, -3n)$, $(3 \cdot 3n, 4 \cdot 3n)$ and finally $(-4 \cdot 3n, -3 \cdot 3n)$ to achieve a total displacement of $(n, 0)$ in 3 moves. Once again, we may similarly do this vertically and hence achieve a displacement of (n, m) for any integers $n, m \neq 0$ in 6 moves. $\square$

Solution 3. *(solution by Gergely Rozgonyi)*

Let the given lattice point be $(N, M) \in \mathbb{Z}^2$. For any integer $n \neq 0$, we consider the two jumps $(4n, -3n)$ and $(-3n, 4n)$ to achieve a total displacement of (n, n) in 2 moves. We may similarly do this in the perpendicular direction and hence achieve a displacement of $(-m, m)$ for some integer $m \neq 0$ in 2 moves.

- Case I: N and M have the same parity: Then the point (N, M) lies on the diagonal line $x + y = N + M$. This line intersects $x = y$ at

$$x = y = \frac{N + M}{2},$$

which is a lattice point due to the parities of N and M. Hence, the (2-step) jump $\left(\frac{N+M}{2}, \frac{N+M}{2}\right)$ followed by the (2-step) jump $\left(\frac{N-M}{2}, -\frac{N-M}{2}\right)$ reaches the given point in at most 4 moves. (Note that any point on the lines $x = y$ and $-x = y$ is attainable in 2 moves.)

- Case II: N and M have different parity: WLOG, assume $N = 2K$, $M = 2L + 1$ for some $K, L \in \mathbb{Z}$. Fredy first jumps $(4, 3)$, reaching $(2(K + 2), 2(L + 2))$. This then reduces to Case I, so any lattice point can be reached in at most 5 moves. $\square$

Problem 3.

(proposed by Fredy Yip)

There are 10^5 users on the social media platform Mathsenger, every pair of which has a direct messaging channel. Prove that each messaging channel may be assigned one of 100 encryption keys, such that no 4 users have the 6 pairwise channels between them all being assigned the same encryption key.

Note: Partial marks will be awarded if the result is proved with the value 100 replaced with 1000 or 10000.

Notes on Marking. Constructions requiring more than 10000 keys are not awarded any marks. Probabilistic arguments giving a construction with 10000 keys are awarded up to 2 marks. Many contestants erroneously believed that $10^5 = 10000$, often leading to worse bounds than expected.

Solution 1.

(solution by Dylan Toh)

We shall show that n encryption keys (colours) may be assigned to messaging channels (edges) between 3^n users (complete graph on 3^n vertices), such that no 4 users have the 6 pairwise channels between them all being assigned the same encryption key (no monochromatic 4-clique). The result then follows by taking $n = 100$, noting that $3^{100} = (3^{20})^5 \geq 10^5$.

Assign each user a distinct ternary length-k string $\overline{a_1 a_2 \dots a_k}$ where $a_1, a_2, \dots, a_k \in \{0, 1, 2\}$. For any pair of users $a = \overline{a_1 \dots a_k}, b = \overline{b_1 \dots b_k}$, assign to their channel any encryption key $i \in \{1, \dots, k\}$ in which their strings differ in the i-th component (i.e. $a_i \neq b_i$). If there are 4 users a, b, c, d with all 6 pairwise channels between them being assigned the same encryption key i, then $a_i, b_i, c_i, d_i \in \{0, 1, 2\}$ are all distinct, a contradiction. □

Solution 2.

(solution by Tony Wang)

Let us borrow the graph theory terminology as described in the previous solution. We will use induction to show that, using n colours, one may edge-colour a complete graph with 3^n vertices so that there are no monochromatic 4-cliques.

- **Base case:** When $n = 1$, the inductive hypothesis is vacuously true.
- **Inductive step:** Suppose that we can edge-colour a complete graph of 3^k vertices using k colours so as to avoid monochromatic 4-cliques. Make three copies of this graph, and colour all the edges that have endpoints in different copies the $(k+1)$-th colour. Now, if we pick any four vertices in the same copy, then their K_4 graph cannot be monochromatic by virtue of the inductive hypothesis. However, if we pick vertices in different copies, then by pigeonhole principle there must also exist two vertices in the same copy, and hence we will have at least one edge coloured using the new colour and at least one edge coloured using one of the old colours. This completes the induction.

As above, we may conclude by noting that $3^{100} = (3^{20})^5 \geq 10^5$, and hence it suffices to choose any subgraph with 10^5 vertices. □

Solution 3. *(solution by David Toh)*

Arrange the users in a circle, and label them in clockwise order as $1, 2, \dots, 10^5$. Denote the distance between users $i, j \in \{1, \dots, 10^5\}$ by

$$d(i,j) = \min(|i-j|, 10^5 - |i-j|).$$

Note $d(i,j) \in \{1, \dots, 50000\}$ for distinct users $i \neq j$. Let $K(i,j)$ be a function mapping pairs of distinct users to the encryption keys $\{1, 2, \dots, 10, A\}$ as follows: for each pair of users (i,j) $(i \neq j)$,

$$K(i,j) = \begin{cases} k, & \text{if} \quad 3^{k-1} \leq d(i,j) < 3^k \quad \text{for some } k = 1, \dots, 9 \\ 10, & \text{if} \quad 3^9 \leq d(i,j) \leq 25000 \\ A, & \text{if} \quad 25000 < d(i,j) \leq 50000. \end{cases}$$

Suppose otherwise that there are 4 users $1 \leq w < x < y < z \leq 10^5$ with the same pairwise encryption keys. Note

$$d(w,x) + d(x,y) + d(y,z) + d(z,w) \leq (x-w) + (y-x) + (z-y) + (10^5 + w - z) = 10^5,$$

so one of the distances above is $\leq 10^5/4 = 25000$. Thus they cannot share encryption key A. Consequently, all 6 pairwise distances are $\in [3^{k-1}, 3^k)$ for some $k = 1, \dots, 10$.

If $z - w \leq 50000$, then

$$3^k > d(z,w) = d(w,x) + d(x,y) + d(y,z) \geq 3^{k-1} + 3^{k-1} + 3^{k-1} = 3^k,$$

a contradiction. Thus $d(z,w) = 10^5 + w - z$ (i.e. the minor arc between z and w along the circle doesn't contain x and y). By rotational symmetry, we must also have $d(w,x) = x - w, d(x,y) = y - x$, and $d(y,z) = z - y$. But summing the distances now gives equality

$$d(w,x) + d(x,y) + d(y,z) + d(z,x) = 10^5,$$

with all distances ≤ 25000 (since the shared encryption key is $k \in \{1, \dots, 10\}$). This implies

$$d(w,x) = d(x,y) = d(y,z) = d(z,x) = 25000,$$

with shared encryption key 10. But then $d(w,y) = 50000 > 25000$, a contradiction. $\square$

Comment. One may also show existence of valid constructions via probabilistic arguments, but these typically require more encryption keys.

Problem 4.

(proposed by Fredy Yip)

Points A, B, C, and D lie on the surface of a sphere with diameter 1. What is the maximum possible volume of tetrahedron $ABCD$?

Notes on Marking. Most solutions went down the route of solution 1, often substituting geometric ideas with analytic computations. In the analytic solutions, points were deducted when maxima arguments were not complete or sufficiently justified. Points were not deducted for simple arithmetic errors in computing the volume of a regular tetrahedron.

Solution 1.

(solution by Tony Wang)

We begin by solving the 2-dimensional version of this problem: given a circle with radius r, what is the largest possible area of a triangle with 3 points on the circle?

To solve this problem, note that if two points A and B are fixed, then the maximal area is obtained when the third point achieves its highest possible altitude from the line AB. In a circle, this is achieved only at the two points where the perpendicular bisector of AB meets the circle. These are also the only two points P where P is equidistant to A and B. By a similar argument on the other two pairs of points, it follows that the triangle does not have maximal area if each point is not equidistant from the other two. This shows that in the 2-dimensional case, the maximal area is achieved by an equilateral triangle. We now prove a useful lemma:

Lemma. The volume of a tetrahedron with base area a and height h is $ah/3$.

Proof. The cross-section of the tetrahedron with a plane of height x yields a shape similar to the base shape, but which has undergone a dilation with dilation factor $(h-x)/h$. Therefore, its area must be $a(h-x)^2/h^2$. Now we can integrate:

$$\int_0^h \frac{a(h-x)^2}{h^2}\,\mathrm{d}x = \left[\frac{ax^3}{3h^2} - \frac{ax^2}{h} + ax + C\right]_0^h = \frac{ah}{3}$$

as desired. $\square$

Now we return to the original three-dimensional problem. Let the plane BCD intersect the sphere at a circle $\mathcal{C}$. The above lemma shows that if BCD does not achieve maximum area on this circle, then the volume of $ABCD$ cannot be maximal. It then follows from the first paragraph that BCD must be equilateral. A symmetric argument holds for all other faces, and hence we have proven that the tetrahedron must be regular.

Now suppose for the sake of contradiction that the centroid H of the tetrahedron is not the centre O of the sphere, then WLOG AH does not pass through O, and we can rotate the tetrahedron and the sphere around AH by $120°$, which has the effect of permuting its vertices while moving O. The tetrahedron has not shifted, but its circumscribing sphere has, a contradiction since any given tetrahedron has at most one circumsphere. Hence, H and O must coincide.

Now it is clear that we must have $AO = BO = CO = DO = 0.5$, and we may now use various methods (some of which are showcased below), to calculate that the volume of the tetrahedron is $\frac{1}{9\sqrt{3}}$. $\square$

Solution 2.

(solution by Dylan Toh)

The volume of $ABCD$ is always bounded above by the volume of the sphere S of radius 0.5. By compactness of the sphere, we may assume $ABCD$ attains the maximum volume.

Let v_1, v_2, v_3, v_4 be vectors in $\mathbb{R}^3$ of length 0.5, representing points $A, B, C, D \in S$ respectively. If Π is the plane through vertices B, C, D, note

$$\text{vol}(ABCD) = \frac{1}{3} \cdot \text{area}(BCD) \cdot (\text{perpendicular height from } A \text{ to } \Pi)$$

which is maximised when $A \in S$ is of maximum distance, i.e. the tangent plane to S at A is parallel to Π. Consequently, v_1 is orthogonal to Π, and thus orthogonal to $\overrightarrow{BC} = v_3 - v_2$; therefore $v_1 \cdot v_2 = v_1 \cdot v_3$ with the Euclidean inner product. Similarly, $v_i \cdot v_j = c$ is constant for all $i \neq j$. Therefore, the tetrahedron $ABCD$ is regular:

$$|AB|^2 = |v_1 - v_2|^2 = |v_1|^2 + |v_2|^2 - 2v_1 \cdot v_2 = 0.5 - 2c$$

and similarly for all other edges. By symmetry of the regular tetrahedron, $v_1 + v_2 + v_3 + v_4 = 0$. Thus

$$0 = v_1 \cdot (v_1 + v_2 + v_3 + v_4) = 0.25 + 3c = 0$$

which implies $c = -\frac{1}{12}$ and the side length of the regular tetrahedron being

$$s = |AB| = \sqrt{0.5 - 2c} = \sqrt{\frac{2}{3}}.$$

By symmetry, we also note that the foot of perpendicular from A to Π is the centre of equilateral triangle BCD, with vector $\frac{1}{3}(v_2 + v_3 + v_4) = -\frac{1}{3}v_1$. The distance from A to Π is thus

$$d(A, \Pi) = \left| v_1 - \left(-\frac{1}{3} v_1 \right) \right| = \frac{4}{3} \cdot 0.5 = \frac{2}{3}.$$

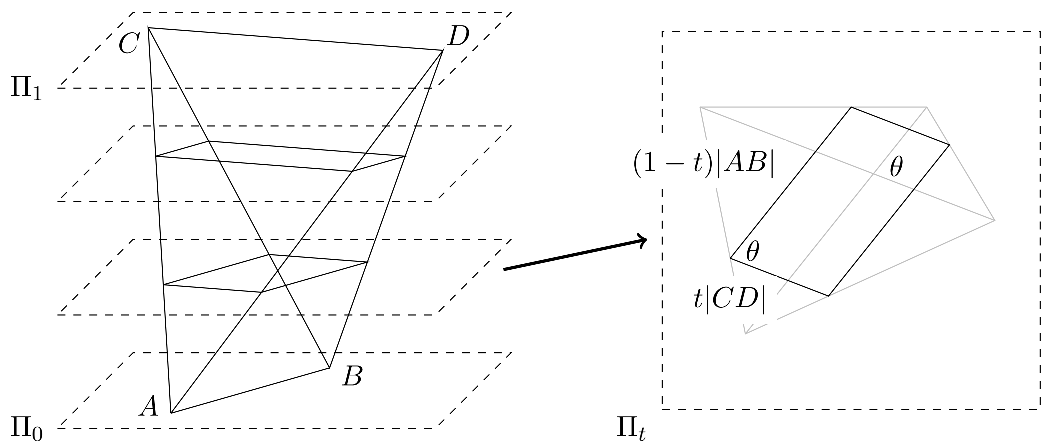
Finally, the maximum volume of a tetrahedron inscribed in S is computed to be

$$\text{vol}(ABCD) = \frac{1}{3} \cdot \text{area}(BCD) \cdot d(A, \Pi) = \frac{1}{3} \cdot \frac{\sqrt{3}}{4} s^2 \cdot \frac{2}{3} = \frac{1}{9\sqrt{3}}. \qquad \square$$

Solution 3.

(solution by Dylan Toh)

Let O be the origin, h be the distance between (infinite) lines AB and CD, Π_0 be a plane through line AB parallel to line CD, $\theta \in [0, \pi/2]$ be the angle between lines AB and CD upon projection onto Π_0, and Π_t be the plane Π_0 translated a distance th towards line CD, for $0 \leq t \leq 1$. We compute the volume of tetrahedron $ABCD$ by taking planar cross-sections Π_t:



In particular, each cross-section $ABCD \cap \Pi_t$ is a parallelogram with side lengths $(1-t)|AB|$ and $t|CD|$, and sides parallel to the respective projections of lines AB and CD to the plane. The tetrahedron volume is then

$$\begin{aligned}\operatorname{vol}(ABCD) &= \int_0^1 \operatorname{area}(ABCD \cap \Pi_t) \cdot h \,\mathrm{d}t \\ &= \int_0^1 (t|AB| \cdot (1-t)|CD| \cdot \sin\theta) \cdot h \,\mathrm{d}t \\ &= \frac{1}{6}|AB||CD|h\sin\theta.\end{aligned}$$

Let 2α be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$, and 2β the angle between $\overrightarrow{OC}$ and $\overrightarrow{OD}$. Then:

- $|AB| = 2r\sin\alpha$, $|CD| = 2r\sin\beta$ (where $r = 0.5$ is the radius of the sphere);
- $h \le d(O, AB) + d(O, CD) = r\cos\alpha + r\cos\beta$; and
- $\sin\theta \le 1$.

Applying standard inequalities, we may tightly bound the tetrahedron volume from above:

$$\begin{aligned}\operatorname{vol}(ABCD) &\le \frac{1}{6} \cdot 2r\sin\alpha \cdot 2r\sin\beta \cdot r(\cos\alpha + \cos\beta) \cdot 1 \\ &\le \frac{2\sqrt{2}r^3}{3} \cdot \sin\alpha \cdot \sin\beta \cdot (\cos^2\alpha + \cos^2\beta)^{1/2} && \text{(QM-AM)} \\ &\le \frac{1}{6\sqrt{2}} \left(\frac{\sin^2\alpha + \sin^2\beta + (\cos^2\alpha + \cos^2\beta)}{3}\right)^{3/2} && \text{(AM-GM)} \\ &= \frac{1}{6\sqrt{2}} \left(\frac{2}{3}\right)^{3/2} = \frac{1}{9\sqrt{3}}.\end{aligned}$$

The bound above is obtained by a regular tetrahedron: this corresponds to values $\alpha = \beta = \cos^{-1}\frac{1}{\sqrt{3}}$ and $\theta = \frac{\pi}{2}$, which clearly corresponds to a possible construction, and satisfies all equality cases above. Alternatively, one may also directly verify equality for an explicit choice of vertices A, B, C, D that form a regular tetrahedron; for instance, the points

$$\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}, \overrightarrow{OD} = \frac{(-1,-1,-1)}{2\sqrt{3}}, \frac{(-1,1,1)}{2\sqrt{3}}, \frac{(1,-1,1)}{2\sqrt{3}}, \frac{(1,1,-1)}{2\sqrt{3}}$$

on the sphere of radius 0.5, are the vertices of a regular tetrahedron of volume

$$\operatorname{vol}(ABCD) = \frac{1}{3!}\left|\det\left(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\right)\right| = \frac{1}{6}\left(\frac{1}{\sqrt{3}}\right)^3 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \frac{1}{9\sqrt{3}}. \qquad \square$$

Problem 5. *(proposed by Dylan Toh)*

(a) Is there a non-linear integer-coefficient polynomial $P(x)$ and an integer N such that all integers greater than N may be written as the greatest common divisor of $P(a)$ and $P(b)$ for positive integers a and b with $a > b$?

(b) Is there a non-linear integer-coefficient polynomial $Q(x)$ and an integer M such that all integers greater than M may be written as $Q(a) - Q(b)$ for positive integers a and b with $a > b$?

Notes on Marking. We decided that part (a) is worth 2 marks, while part (b) yielded the remaining 8 marks. No marks were awarded for the correct answer to either part. 1 partial mark could be obtained in part (a) by considering a working polynomial. In part (b), both case I and case II were worth 4 marks. 1 partial mark was awarded for proving every large enough prime was of the form $Q(x+1) - Q(x)$. Solutions to case I by counting which were off by a constant factor achieved 3 marks. 2 marks were available in case II for reducing to the leading coefficient being either 1 or 2.

Solution 1. *(solution by Dylan Toh)*

(a) The answer is **yes**. For example, let $P(x) = x(2x+1)$. Then for all positive integers n,

$$\gcd(P(2n), P(n)) = n \cdot \gcd(2n+1, 8n+2) = n \cdot \gcd(2n+1, 2) = n.$$

(b) The answer is **no**. To prove this, suppose for the sake of contradiction that there exists such a polynomial Q. Now, consider separately the cases where $\deg(Q) \geq 3$ and $\deg(Q) = 2$.

- Case I: $\deg Q \geq 3$. The key idea for this case is that Q grows too rapidly to admit all positive integers as differences.

  First, note that the condition implies that $Q(x)$ takes arbitrarily large positive values for positive x. This implies that the leading coefficient of Q is positive, and $Q(x)$ is eventually increasing, with $Q(x) \to \infty$ as $x \to \infty$. In particular, there is a fixed integer N_0 such that $Q(x) > Q(y)$ for all $x > y > 0$ and $x \geq N_0$.

  Hence, any difference $k = Q(a) - Q(b)$ with $a > N_0$ satisfies

  $$k = Q(a) - Q(b) \geq Q(a) - Q(a-1) = R(a),$$

  where $R(x) = Q(x) - Q(x-1)$ is an integer-coefficient polynomial with leading coefficient $\geq \deg Q$, and $\deg R = \deg Q - 1 \geq 2$. Since $R(x) \geq x^2$ for all sufficiently large x, there is a positive constant C such that $R(x) \geq x^2 - C$ for all $x \geq N_0$. Consequently, we can write $k \geq a^2 - C$, or $a \leq \sqrt{k+C}$.

  We finish with a count: suppose all positive integers larger than M can be expressed as $Q(a) - Q(b)$ for positive integers $a > b$. WLOG, we may assume that M is larger than any $Q(x) - Q(y)$ where $N_0 \geq x > y > 0$. Then for each difference $k = Q(a) - Q(b) > M$, we must have $a > N_0 \implies a \leq \sqrt{k+C}$. Now consider the set of numbers $\{M+1, M+2, \ldots, N\}$, for some N, which we wish to express as differences $Q(a) - Q(b)$ for $a > b > 0$. However, since $a \leq \sqrt{N+C}$, the number of distinct pairs $(a,b) \in \mathbb{Z}^2$ with $\sqrt{N+C} \geq a > b > 0$ is just $\binom{\lfloor\sqrt{N+C}\rfloor}{2} \leq \frac{N+C}{2}$. So we want to make $N - M$ differences, but we can only make at most $(N+C)/2$ of them. Setting $N = 2M + 2C$ shows that we cannot make all the differences we want, finishing the contradiction.

- Case II: $\deg Q = 2$. Let $Q(x) = ax^2 + bx + c$. The key idea for this case is to resolve the parity of coefficients, then work modulo $2a$. For all $m, n \in \mathbb{Z}$ with $m > n$,

$$\begin{aligned} Q(m) - Q(n) &= a(m^2 - n^2) + b(m - n) \\ &= (m - n)(a(m + n) + b). \end{aligned}$$

Pick a sufficiently large prime $p = p \cdot 1$ with $p \equiv 1 \pmod{2a}$, and express it as a difference $p = Q(n+d) - Q(n) = d(a(2n+d) + b)$ where $d > 0$. This implies each factor d and $a(2n+d)+b$ is $\equiv 1 \pmod{2a}$. In particular, $a + b$ is odd.

Now, express $2p = 2p \cdot 1 = p \cdot 2$ as a difference $2p = Q(n' + d') - Q(n') = d'(a(2n' + d') + b)$ where $d' > 0$. This implies one of the factors d' and $a(2n' + d') + b$ is $\equiv 2 \pmod{2a}$, while the other is $\equiv 1 \pmod{2a}$. Thus

$$\begin{aligned} 1 &\equiv d' + a(2n' + d') + b \\ &\equiv d' + ad' + a + 1 \\ &\equiv (a+1)(d'+1) \pmod{2}, \end{aligned}$$

so a is even and b is odd; and the factors $d' \equiv 2 \pmod{2a}$ and $a(2n' + d') + b \equiv 1 \pmod{2a}$. Consequently, $b \equiv 1 \pmod{2a}$.

Finally, we return to the expression for p, and note

$$\begin{aligned} 1 &\equiv a(2n + d) + b \\ &\equiv ad + 1 \\ &\equiv a + 1 \pmod{2a}, \end{aligned}$$

a contradiction. □

Solution 2.

(solution by Tony Wang)

We present an alternative method for Case II of part (b) above.

- Case II: $\deg Q = 2$. Let $Q(x) = ax^2 + bx + c$. WLOG, we may assume that $c = 0$ and that a is positive. Now, note that a and b cannot be both odd or both even, as otherwise $Q(x)$ will always be even. If a is odd and b is even, then a quick check reveals that $Q(x) - Q(y)$ can never be 2 mod 4. Hence, a must be even and b must be odd. Let $a = 2d$ and $b = 2e + 1$ for positive integers d and e.

  Now, $Q(x) = 2dx^2 + (2e+1)x$, and so $Q(x) - Q(y) = (2d(x+y) + 2e + 1)(x - y)$. What is the smallest number of this form that is divisible by 2^n for some n? Since the first factor is always odd, the smallest value must be achieved when $x = 2^n + 1$ and $y = 1$, at which point $Q(x) - Q(y) > 2^{2n}$. Because no smaller number divisible by 2^n can be written as $Q(x) - Q(y)$, this proves that there are arbitrarily large inexpressible integers, a contradiction. □

Solution 3.

(solution by contestants)

We present a different approach to both cases of part (b). Note that $a - b \mid Q(a) - Q(b)$. Assuming that $p = Q(a_p) - Q(b_p)$, this gives

$$a_p - b_p \mid p \implies a_p - b_p = 1, p.$$

However, the latter case can only happen for finitely many p. Indeed, if $a_p - b_p = p = Q(a_p) - Q(b_p)$, then consider the polynomial $R(x) = Q(x) - x$. We have $R(a_p) = R(b_p)$.

However, R is a polynomial of degree at least 2 with positive leading coefficient, hence for some positive integer T we must have $R(x)$ strictly increasing for $x \geq T$ and $R(T)$ greater than $\sup_{x\in[0,T]} R(x)$.

Hence if $a_p > b_p$ and $a_p > T$, then we cannot have $R(a_p) = R(b_p)$. Consequently the primes for which $a_p - b_p = p$ must obey $b_p < a_p \leq T$, which only covers finitely many primes.

Hence, all large enough primes p are of the form $Q(x+1) - Q(x) = S(x)$. Now we can finish the problem off by breaking into cases, as in the previous solutions.

- Case I: $\deg Q \geq 3$ In this case S is a polynomial of degree $d \geq 2$ which takes every prime number.

  One can continue this argument in a number of ways. Contestants successfully argued that as the image of S took $\mathcal{O}(n^{1/d})$ values less than n, and the number of primes less than n is $\mathcal{O}(n/\log n)$ by prime number theorem, that this was impossible.

  Others appealed to the boundedness of prime gaps: the gaps between successive elements in the image of S goes to infinity, however as proven by Zhang, there are infinitely many pairs of primes with difference at most 70000000.

  One solution by Tejas Mittal of Oxford considered $\sum \frac{1}{S(n)}$ for large n. As this sum contains all large enough prime numbers, it must diverge. However asymptotically it is $\sum \frac{1}{n^2}$, which converges.

- Case II: $\deg Q = 2$. Let $Q(x) = ax^2 + bx + c$. Then $S(x) = 2ax + (a+b)$.

  Hence every large enough prime is of the form $2ax + (a+b) \equiv a + b \pmod{2a}$. By Dirichlet's theorem, there are infinitely many primes congruent to m modulo $2a$ for every $(m, 2a) = 1$. This gives a contradiction unless $a = 1$.

  Now all large primes are $b + 1 \pmod 2$, hence b is even. But if $b = 2k$, then $Q(x) = (x+k)^2 - d$, hence every large enough integer must be the difference of two squares. But we cannot achieve any integers 2 mod 4 this way, our final contradiction. $\square$

Solution 4.

(solution by Ishan Nath)

We give an alternative proof for case II of the above solution, without appealing to Dirichlet's.

- Case II: $\deg Q = 2$. Again let $Q(x) = ax^2 + bx + c$.

  Notice that $a + b$ must be odd in order to obtain odd integers. Consider a large prime p, then let $p = Q(r+1) - Q(r) = 2ar + (a+b)$.

  Now consider $2p = Q(s+t) - Q(s)$. Again notice $t \mid 2p$, so $t = 1, 2, p, 2p$. Only finitely many primes can satisfy $t = p$, by a similar argument to solution 3 but looking at the polynomial $Q(x) - 2x$, so either $t = 1$ or 2 for large p.

  But we cannot have $t = 1$ otherwise $2p = 2as + (a+b)$ is odd, hence $t = 2$ and $2p = 4as + (4a + 2b)$.

  But now $4ar + (2a+2b) = 2p = 4as + (4a+2b)$, hence $4a(r-s) = 2a$, which is impossible for integers r, s. $\square$

Comment. In Case I of part (b), it is not necessary to restrict m and n to be positive integers: with some care, one may account for the values of $Q(x)$ for large negative values of x. However, this restriction is crucial for Part II, as otherwise the polynomial $Q(x) = x(2x+1)$ will suffice.

Problem 6.

(proposed by Dylan Toh)

Let $f : \mathbb{N} \to \mathbb{N}$ be a bijection of the positive integers. Prove that, as $N \to +\infty$, at least one of the limits

$$\sum_{n=1}^{N} \frac{1}{n + f(n)} \to +\infty \qquad \text{or} \qquad \sum_{n=1}^{N} \frac{f(n) - n}{nf(n)} \to +\infty$$

is true.

Note: The function $f : \mathbb{N} \to \mathbb{N}$ is a bijection if, for every positive integer a, there is a unique positive integer n such that $f(n) = a$.

Notes on Marking. 2 marks were awarded for setting up a formal notion of density $d(N)$, or a quantity of equivalent strength. 3 marks were awarded for the full Case I below, with 1 partial mark awarded for the idea of using the divergence of harmonic series to bound the former sum in this case (no marks were awarded for observing the divergence of harmonic series outside this case). 5 marks were awarded for the full Case II below, with 1 partial mark awarded for the idea of using the logarithm to estimate the harmonic series and bound the latter sum in this case. No marks were awarded for considering a linear combination of the sums.

Solution 1.

(solution by Dylan Toh)

Note that the former limit

$$A = \sum_{n=1}^{\infty} \frac{1}{n + f(n)}$$

is a sum of non-negative terms, thus it converges to the supremum of partial sums. Meanwhile, the latter limit

$$B = \sum_{n=1}^{\infty} \frac{f(n) - n}{nf(n)} = \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{f(n)} \right) = \lim_{N\to\infty} B_N$$

need not exist in general, although all partial sums B_N are non-negative, since the inequality

$$\sum_{n=1}^{N} \frac{1}{n} \geq \sum_{n=1}^{N} \frac{1}{f(n)}$$

is clear (say, by reordering $f(1), \ldots, f(N)$ to increasing order, then comparing term-by-term).

For each positive integer N, define

$$d(N) = \frac{1}{N} \cdot \#\{\sqrt{N} \leq n \leq N : f(n) \leq N\} \in [0, 1].$$

This captures the density of small values of f among the first N terms. Intuitively, $d(N)$ being non-zero for many large N will cause A to diverge, while $d(N)$ eventually tending to zero should lead to the divergence of B. We formalise this in the following two cases:

- Case I: $d(N) \not\to 0$ as $N \to \infty$. Thus there is $\varepsilon > 0$ and infinitely many N such that $d(N) \geq \varepsilon$. For each such N, note

$$\sum_{n=\lceil\sqrt{N}\rceil}^{N} \frac{1}{n + f(n)} \geq Nd(N) \cdot \frac{1}{2N} \geq \frac{\varepsilon}{2}.$$

Therefore, one may pick $N_1, N_2, N_3, \dots$ with $\sqrt{N}_1 < N_1 < \sqrt{N}_2 < N_2 < \sqrt{N}_3 < N_3 < \dots$ such that $d(N_k) \geq \varepsilon$ for all k; consequently,

$$A \geq \sum_{k=1}^{\infty} \sum_{\sqrt{N}_k \leq n \leq N_k} \frac{1}{n + f(n)} \geq \sum_{k=1}^{\infty} \frac{\varepsilon}{2} = +\infty.$$

- Case II: $d(N) \to 0$ as $N \to \infty$. Thus for all $\varepsilon > 0$ (WLOG $\varepsilon < \frac{1}{2}$), there is N_0 such that $d(N) \leq \varepsilon$ for all $N \geq N_0$. WLOG increase N_0 such that $\sqrt{N}_0 > \varepsilon^{-1}$. Thus for all $N \geq N_0$,

$$\#\{n \leq N : f(n) \leq N\} \leq \sqrt{N} + N d(N) \leq 2\varepsilon N$$

and thus

$$\begin{aligned} B_N = \sum_{n=1}^{N} \frac{1}{n} - \sum_{n=1}^{N} \frac{1}{f(n)} &\geq \sum_{n=1}^{N} \frac{1}{n} - \sum_{n=1}^{\lfloor 2\varepsilon N \rfloor} \frac{1}{n} - (N - \lfloor 2\epsilon N \rfloor)\frac{1}{N} \\ &\geq -1 + \sum_{2\epsilon N < n \leq N} \frac{1}{n} \\ &\geq -2 + \int_{2\varepsilon N}^{N} \frac{dx}{x} = -2 + \log\bigl(\varepsilon^{-1}/2\bigr). \end{aligned}$$

Finally, noting that $-2 + \log\bigl(\varepsilon^{-1}/2\bigr) \to +\infty$ as $\varepsilon \to 0$, we conclude $B_N \to +\infty$ as $N \to \infty$. $\square$

Comment. There are scenarios in which one series diverges while the other converges. For instance, defining f to swap the k-th power of 2 with the k-th non-power of 2, the former sum A is bounded above by $2\sum_{k=0}^{\infty} 2^{-k} = 4$. On the other hand, one may iteratively define f to make B converge to any non-negative real number: for instance, if we wish $B_N \to c$ as $N \to \infty$, we may iteratively define

$$f(N+1) = \begin{cases} \min(\mathbb{N} \setminus \{f(1), \dots, f(N)\}), & B_N > c \\ 2^N, & B_N \leq c. \end{cases}$$