IMPERIAL-CAMBRIDGE
MATHEMATICS
COMPETITION

ROUND TWO

25 February 2024

Name: ______________________________

Contestant ID: ______________________________

University: ______________________________

Instructions:

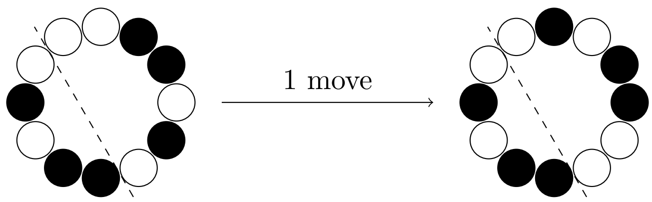
- Do not turn over until told to do so.
- You will have 4 hours to solve 5 problems, each of which carries 10 marks.
- Use a black or blue pen or a dark pencil. Rulers, compasses, protractors, and erasers may be used but will not be required. All electronic devices, including calculators, are prohibited.
- Drinks are allowed, but food is prohibited.
- Write your solution to each problem on a different page. At the top of each page, write down the question number, your initials, and your contestant number. Use both sides whenever possible. Writing clearly and not too faintly – your work will be scanned for marking.
- Problems are listed roughly in order of difficulty. Proofs are expected for all problems even if they only ask for an answer.
- One complete solution will be awarded more marks than several unfinished attempts.
- You may not leave the contest venue in the first two hours or the last thirty minutes unless exceptional circumstances arise.
- You may take away the problems sheet and any rough work when leaving the venue.

Problem 1.

(a) Prove that there exist distinct positive integers $a_1, a_2, \ldots, a_{2024}$ such that for each $i \in \{1, 2, \ldots, 2024\}$, a_i divides $a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_{2024} + 1$.

(b) Prove that there exist distinct positive integers $b_1, b_2, \ldots, b_{2024}$ such that for each $i \in \{1, 2, \ldots, 2024\}$, b_i divides $b_1 b_2 \cdots b_{i-1} b_{i+1} \cdots b_{2024} + 2024$.

Problem 2. Let $n \geq 3$ be a positive integer. A circular necklace is called *fun* if it has n black beads and n white beads. A *move* consists of cutting out a segment of consecutive beads and reattaching it in reverse. Prove that it is possible to change any fun necklace into any other fun necklace using at most $(n-1)$ moves.



Note: Rotations and reflections of a necklace are considered the same necklace.

Problem 3. Let N be a fixed positive integer, S be the set $\{1, 2, \ldots, N\}$, and F be the set of functions $f : S \to S$ such that $f(i) \geq i$ for all $i \in S$. For each $f \in F$, let P_f be the unique polynomial of degree less than N satisfying $P_f(i) = f(i)$ for all $i \in S$.

If f is chosen uniformly at random from F, determine the expected value of $(P_f)'(0)$, where

$$(P_f)'(0) = \left. \frac{\mathrm{d}P_f(x)}{\mathrm{d}x} \right|_{x=0}.$$

Problem 4. Let $(t_n)_{n \geq 1}$ be the sequence defined recursively by $t_1 = 1$, $t_{2k} = -t_k$, and $t_{2k+1} = t_{k+1}$ for all $k \geq 1$. Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{t_n}{\sqrt[2024]{n}}.$$

(a) Prove that the series converges to a real number c.

(b) Prove that c is non-negative.

(c) Prove that c is strictly positive.

Problem 5. Is it possible to dissect an equilateral triangle into 3 congruent polygonal pieces (not necessarily convex), one of which contains the triangle's centre in its interior?

Note: The interior of a polygon does not include its perimeter.