IMPERIAL-CAMBRIDGE
MATHEMATICS
COMPETITION

7th Edition (2023–2024)

ROUND TWO

Official Solutions*

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*A solution may receive full or partial marks even if it does not appear in this booklet.

Problem 1.

(proposed by Ishan Nath)

(a) Prove that there exist distinct positive integers $a_1, a_2, \ldots, a_{2024}$ such that for each $i \in \{1, 2, \ldots, 2024\}$, a_i divides $a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_{2024} + 1$.

(b) Prove that there exist distinct positive integers $b_1, b_2, \ldots, b_{2024}$ such that for each $i \in \{1, 2, \ldots, 2024\}$, b_i divides $b_1 b_2 \cdots b_{i-1} b_{i+1} \cdots b_{2024} + 2024$.

Notes on Marking. Most attempts were constructive. 1 mark was awarded for constructing a (possibly invalid) sequence inductively. 4 marks were awarded for a valid construction for either part. 2 additional marks were awarded for a valid construction for both parts. Proving the construction for part (a) was valid was worth 1 mark, while proving the construction for part (b) was valid was worth 2 marks. 1 mark was deducted if the construction was not trivially increasing and the elements of the sequence were not proven to be distinct.

Solution 1.

(solution by Dylan Toh)

We show that such numbers exist by construction:

(a) Set $a_1 = 1$ and $a_i = a_1 a_2 \cdots a_{i-1} + 1$ for $i = 2, 3, \ldots, 2024$. This construction satisfies $a_j \equiv 1 \pmod{a_i}$ for all $j > i \geq 1$, so the divisor relation is satisfied for all $i \geq 2$:

$$a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_{2024} + 1 \equiv a_1 a_2 \cdots a_{i-1} + 1 = a_i \equiv 0 \pmod{a_i}, \quad i \geq 2$$

and the divisor relation is trivially true for $i = 1$ (since $a_1 = 1$ divides any positive integer).

(b) Set $a_1 = 2024, a_2 = 4048$, and $a_i = a_2 a_3 \ldots a_{i-1} + 1$ for $i = 3, \ldots, 2024$. This construction satisfies $a_j \equiv 1 \pmod{a_i}$ for all $j > i \geq 2$, so the divisor relation is satisfied for all $i \geq 2$:

$$\begin{aligned} a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_{2024} + 2024 &\equiv a_1 a_2 \cdots a_{i-1} + 2024 \\ &\equiv \begin{cases} 2024 a_i \equiv 0 \pmod{a_i}, & i \geq 3 \\ a_1 + 2024 \equiv 0 \pmod{a_i}, & i = 2. \end{cases} \end{aligned}$$

For $i = 1$, note $a_2 \cdots a_{2024} + 2024 = a_1(2a_3 \cdots a_{2024} + 1)$ is a multiple of $a_1 = 2024$. □

Solution 2.

(solution by contestants)

We show a different construction for part (b) (using a_i from part (a) of the previous solution):

(b) Set $b_i = a_i$ for $i = 1, 2, \ldots, 2023$ and $b_{2024} = b_1 b_2 \ldots b_{2023} + 2024$. This construction now satisfies $b_j \equiv 1 \pmod{b_i}$ for all $2023 \geq j > i \geq 1$, so for all $2023 \geq i \geq 2$,

$$b_1 \ldots b_{i-1} \underbrace{b_{i+1} \ldots b_{2023}}_{\equiv 1 \pmod{b_i}} b_{2024} + 2024 \equiv 2024(b_1 \ldots b_{i-1}) + 2024 = 2024 b_i \equiv 0 \pmod{b_i},$$

and for $i = 2024$ we have

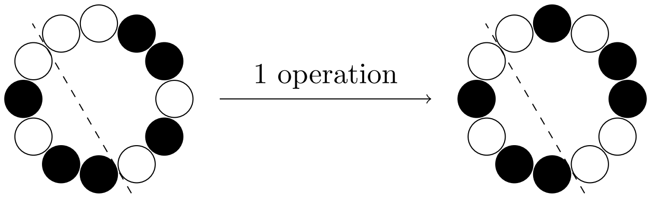
$$b_1 b_2 \ldots b_{2023} + 2024 = b_{2024} \equiv 0 \pmod{b_{2024}}.$$

The $i = 1$ case is trivially true. □

Problem 2.

(proposed by Dylan Toh)

Let $n \geq 3$ be a positive integer. A circular necklace is called *fun* if it has n black beads and n white beads. A *move* consists of cutting out a segment of consecutive beads and reattaching it in reverse. Prove that it is possible to change any fun necklace into any other fun necklace using at most $(n-1)$ moves.



Note: Necklaces related by rotations or reflections are the same necklace.

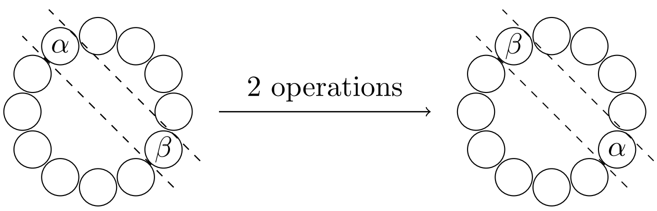
Notes on Marking. Many contestants attempted induction on the question statement as given, and were unsuccessful at proving the induction step due to the lack of control over the restored positions of the removed/merged beads in the original necklaces. Other unsuccessful attempts were typically because the proposed algorithm failed to preserve the agreement in the necklaces established by previous moves.

No marks were awarded for attempts to induct directly on the question statement, or for using more moves (e.g. $2n$ moves). 2 partial marks were awarded for any of the first key steps of the solutions below: (Solution 1) swapping 2 beads in 2 moves, (Solution 2) induction on a string of beads, (Solution 3) induction on the length of a maximal common substring, or (Solution 4) transforming any necklace into the fully ordered or fully alternating necklace. Additional marks were awarded for further ideas or partial resolution of cases involved in the induction steps.

Solution 1.

(solution by Dylan Toh)

The key observation for this solution is that 2 operations may be performed to precisely swap a pair of beads in a necklace while preserving the rest of the necklace:



Denote a necklace by a sequence $a = (a_1, \ldots, a_{2n})$, where colours black and white are represented by $+1$ and -1 respectively, and $a_i \in \{\pm 1\}$ is the colour of the i-th bead in anticlockwise order. Note that necklaces are equivalent up to rotations $(a_1, a_2, \ldots, a_{2n-1}, a_{2n}) \sim (a_2, a_3, \ldots, a_{2n}, a_1)$ and reflections $(a_1, a_2, \ldots, a_{2n-1}, a_{2n}) \sim (a_{2n}, a_{2n-1}, \ldots, a_2, a_1)$.

Let a and b be two configurations of necklaces. Comparing a random rotation of a with b, each bead of b agrees with the corresponding bead of a exactly half the time, so the average number of beads agreed upon is exactly $\frac{1}{2} \cdot 2n = n$. Formally, taking indices modulo $2n$,

$$X_j = \sum_{i=1}^{2n} \mathbb{1}_{a_{i+j}=b_i} = \#\text{ beads agreed upon by } (a \text{ rotated clockwise by } j \text{ beads}) \text{ and } b$$

takes the average value

$$\frac{1}{2n}\sum_{j=1}^{2n} X_j = \frac{1}{2n}\sum_{i,j=1}^{2n} \mathbb{1}_{a_{i+j}=b_i} = \sum_{i,k=1}^{2n} \mathbb{1}_{a_k=b_i} = \frac{1}{2n}\cdot 2n^2 = n.$$

Therefore, $X_j \geq n$ for some j. Furthermore, each X_j is even, since

$$X_j \equiv \sum_{i=1}^{2n}(\mathbb{1}_{a_{i+j}=1} + \mathbb{1}_{b_i=1}) = \sum_{k=1}^{2n} \mathbb{1}_{a_k=1} + \sum_{i=1}^{2n} \mathbb{1}_{b_i=1} = n + n \equiv 0 \pmod 2.$$

We now split cases by the parity of n:

- Case I: n odd. WLOG $X_0 \geq n$. Since X_0 is even, thus $X_0 \geq n+1$. Thus a and b differ in $\leq n-1$ beads, and may be made equal via $\leq \frac{n-1}{2}$ pairwise swaps, i.e. via $\leq 2\cdot\frac{n-1}{2} = n-1$ operations.
- Case II(a): n even, $X_j > n$ for some j. WLOG $X_0 \geq n+2$. Then a and b differ in $\leq n-2$ beads, and may be made equal via $\leq \frac{n-2}{2}$ pairwise swaps, i.e. via $\leq 2\cdot\frac{n-2}{2} = n-2$ operations.
- Case II(b): n even, $X_j = n$ for all j. Note that some pair of adjacent beads in a is black followed by white, while some pair of adjacent beads in b is white followed by black. We may thus WLOG (by rotating a and b) that $a_1 = b_2 = +1$, $a_2 = b_1 = -1$. Since $X_0 = n$, thus reversing the segment a_1a_2 with a single operation makes $X_0 = n+2$. By Case II(a), the necklaces may then be made equal by a further $n-2$ operations. □

Solution 2.

(solution by Dylan Toh)

Fix a black bead on each necklace, and align the necklaces with respect to the fixed black bead. Each necklace may then be viewed as an ordered string of $n-1$ black and n white beads. It suffices to prove the following statement, by induction on k: given two strings of beads of equal length l, each having k black beads, they are related by at most $k-1$ operations (where an operation consists of reversing a consecutive substring of beads).

The statement is trivially true for $k = 0$ (such strings are entirely white). For $k \geq 1$, let $i, j \in \{1, \ldots, l\}$ be the respective positions of the leftmost black beads of the two strings; WLOG $i \geq j$. Reversing the substring $\{j, \ldots, i\}$ of the first string results in both strings having the leftmost black bead in position j. We then apply the induction hypothesis to make the remaining substrings $\{j+1, \ldots, l\}$ equal upon a further $k-2$ operations. □

Solution 3.

(solution by contestants)

We show that if two aligned necklaces agree on a string of k beads, then then one may perform a single move (or not perform a move at all) to extend the agreement to $k+2$ beads. The result then follows from induction, beginning at $k = 2$ with no moves: one may pick a pair of adjacent beads of opposite colour in each necklace, then rotate/reflect the necklaces such that they are aligned.

Label the beads beside the agreed string for each necklace:

$$\ldots ab[\text{agreed string}]cd \ldots \qquad \ldots ef[\text{agreed string}]gh \ldots$$

where $a, b, c, d, e, f, g, h \in \{\pm 1\}$ and WLOG $b = +1$. By extending the agreed string if possible, one may also assume $b \neq f$ and $c \neq g$. Let there be m_+ black and m_- white beads in the segment of each necklace excluding the agreed string; these numbers must agree on both strings, since both have the same number of black and white beads. We then consider a series of cases:

1. If $a \neq b$, then $m_+, m_- \geq 1$. Here, have $a = -1, b = +1, f = -1$. Let $x = +1$ be the first black bead on the left of the agreed string on the second necklace. One may reverse the string $x \ldots ef$ on the second necklace to get further agreement on the two beads left of the original agreed string.

2. The cases $c \neq d$, $e \neq f$, or $g \neq h$ are similarly resolved.

3. If $c \neq b$, then $c = -1$, then $b = g = +1$ and $c = f = -1$. One may thus reverse the string $ba \ldots dc$ on the first necklace, to get further agreement on the beads left and right of the original agreed string.

4. The remaining case is $a = b = c = d = +1$, $e = f = g = h = -1$. WLOG $m_+ \geq m_-$. If within the remaining portion of the second necklace, each black bead has a white bead to its left, then one may carry out this pairing, in which some white beads remain unpaired (such as f and g), implying $m_+ < m_-$, a contradiction. Thus there must be a pair of consecutive black beads xy ($x = y = +1$) in the remaining portion of the second necklace. One may reverse the string $xy \ldots ef$ of the second necklace to get further agreement on the two beads left of the original agreed string.

Since $2n = 2 + 2 \cdot (n-1)$, thus full agreement can be achieved in at most $n - 1$ moves. □

Solution 4. *(solution by contestants)*

For necklaces A, B, let $d(A, B)$ denote the minimum number of moves required to change one necklace into the other. We note that this defines a discrete metric on the set of necklaces: in particular, the triangle inequality $d(A,B) + d(B,C) \geq d(A,C)$ holds true, by concatenating the sequences of moves.

For a necklace A, let $m(A)$ denote the number of pairs of adjacent beads differing in colour. Let Ω be the necklace with n consecutive white beads followed by n consecutive black beads, and Π the fully alternating necklace; note $2 \leq m(A) \leq 2n$, equality on the left iff $A = \Omega$, and equality on the right iff $A = \Pi$. Furthermore, $m(A)$ is always even: it counts the number of contiguous monochromatic blocks, and the contiguous blocks alternate in colour around the necklace.

First, we show for any necklace A with $m(A) > 2$, there is a move that reduces $m(A)$ by 2. Pick two pairs of adjacent beads ab, cd with the black bead on the left of the white bead (i.e. $a = c = +1, b = d = -1$); this is possible since there are at least two contiguous blocks of white beads, and one may take each of their left boundaries. Then the move

$$\ldots a|b \ldots c|d \ldots \quad \mapsto \quad \ldots a|c \ldots b|d \ldots$$

reduces $m(A)$ by 2.

Next, we show that for any necklace A with $m(A) < 2n$, there is a move that increases $m(A)$ by 2. Pick a pair of adjacent white beads ab; this is always possible, otherwise each white bead may be paired with the black bead to the right of it, and the necklace formed by these n pairs is fully alternating. Similarly, pick a pair of adjacent black beads cd. Then, the move

$$\ldots a|b \ldots c|d \ldots \quad \mapsto \quad \ldots a|c \ldots b|d \ldots$$

increases $m(A)$ by 2.

Finally, we note that if $m(A) = 2k$, then by repeatedly performing moves to decrease $m(A)$, one has $d(A, \Omega) \leq k - 1$, while by repeatedly performing moves to increase $m(A)$, one has $d(A, \Pi) \leq n - k$. Therefore, $d(A, \Omega) + d(A, \Pi) \leq n - 1$.

Finally, for two necklaces A, B, one applies the triangle inequality to obtain

$$2d(A,B) \leq d(A,\Omega) + d(\Omega,B) + d(A,\Pi) + d(\Pi,B) \leq 2(n-1),$$

thus $d(A,B) \leq n-1$, so one may transform A to B in at most $n - 1$ moves. □

Comment. The number of operations $n-1$ is tight. This may be seen by noting (in the notation of Solution 4) that a single move on a necklace A changes $m(A)$ by at most 2. Therefore, at least $n-1$ operations are required to go between necklaces Ω and Π.

Problem 3.
(proposed by Ishan Nath)

Let N be a fixed positive integer, S be the set $\{1, 2, \ldots, N\}$, and F be the set of functions $f : S \to S$ such that $f(i) \geq i$ for all $i \in S$. For each $f \in F$, let P_f be the unique polynomial of degree less than N satisfying $P_f(i) = f(i)$ for all $i \in S$.

If f is chosen uniformly at random from F, determine the expected value of $(P_f)'(0)$, where

$$(P_f)'(0) = \left.\frac{\mathrm{d}P_f(x)}{\mathrm{d}x}\right|_{x=0}.$$

Notes on Marking. No marks were deducted for omitting the $N = 1$ case. 2 partial marks were awarded for formulating a correct summation that evaluated to the answer, as in solution 1. Additional marks were awarded for evaluating specific terms of this summation.

No marks were obtained for observing this was the average value of the x coefficient in each polynomial. Setting up or establishing the fact that the solution was linear in the space of polynomials was worth up to 5 marks.

Solution 1.
(solution by Dylan Toh)

For $N = 1$, the polynomial is constant, so the answer is 0. Henceforth, we assume $N \geq 2$.

Identifying $f \in S$ with the tuple $(f(1), f(2), \ldots, f(N)) \in \mathbb{R}^N$, we equivalently sample f uniformly among the finite set $\{1, \ldots, N\} \times \{2, \ldots, N\} \times \cdots \times \{N\}$ of $N!$ tuples in $\mathbb{R}^N$.

Note the Lagrange interpolation map $\Psi : \mathbb{R}^N \to \mathbb{R}[x]$ given by

$$a = (a_1, \ldots, a_N) \stackrel{\Psi}{\mapsto} \sum_{i=1}^{N} a_i \left(\prod_{j \in \{1,\ldots,N\} \setminus \{i\}} \frac{x - j}{i - j} \right)$$

is linear as a map between $\mathbb{R}$-vector spaces, and $\Psi(a)$ is the unique polynomial P of degree $< N$ such that $P(i) = a_i$ for $i = 1, \ldots, N$. Also, the derivative-evaluation map $\left.\frac{d}{dx}\right|_{x=0} : \mathbb{R}[x] \to \mathbb{R}$ is linear. Thus by linearity of expectation,

$$\text{Ans} = \mathbb{E}\left[\left.\frac{dP_f(x)}{dx}\right|_{x=0} \right] = \left.\frac{d}{dx}\right|_{x=0} \mathbb{E}\left[\Psi(f)\right] = \left.\frac{d}{dx}\right|_{x=0} \Psi(\mathbb{E}[f]).$$

$\mathbb{E}[f]$ may be computed conveniently, again by linearity of expectation in each coordinate:

$$\mathbb{E}[f] = \left(\frac{1 + 2 + \cdots + N}{N}, \frac{2 + \cdots + N}{N - 1}, \ldots, \frac{N}{1} \right) = \frac{1}{2}(N + 1, N + 2, \ldots, N + N).$$

Thus $\Psi(\mathbb{E}[f]) = \frac{1}{2}(N + x)$, and the desired answer is $\left.\frac{d}{dx}\right|_{x=0} \frac{N+x}{2} = \frac{1}{2}$. $\square$

Solution 2.
(solution by contestants)

Using Lagrange interpolation we may write

$$P_f(x) = \sum_{i=1}^{N} f(i) \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{x - j}{i - j} = \sum_{i=1}^{N} \frac{(x - 1) \cdots \widehat{(x - i)} \cdots (x - N)}{(i - 1) \cdots \widehat{(i - i)} \cdots (i - N)} f(i),$$

where the hat indicates the corresponding term is not present in the product. From this the derivative may be computed:

$$(P_f)'(x) = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{(x-1)\cdots\widehat{(x-i)}\cdots\widehat{(x-j)}\cdots(x-N)}{(i-1)\cdots\widehat{(i-i)}\cdots(i-N)} f(i),$$

$$\begin{aligned}
(P_f)'(0) &= \sum_{i \neq j} \frac{(-1)\cdots\widehat{(-i)}\cdots\widehat{(-j)}\cdots(-N)}{(i-1)\cdots\widehat{(i-i)}\cdots(i-N)} f(i) \\
&= \sum_{i \neq j} \frac{N!(-1)^N}{(i-1)!(N-i)!(-1)^{N-i} \cdot i \cdot j} f(i) \\
&= \sum_{i \neq j} \frac{(-1)^i}{j} \binom{N}{i} f(i) = \sum_{i,j} \frac{(-1)^i}{j} \binom{N}{i} f(i) - \sum_{i=j} \frac{(-1)^i}{j} \binom{N}{i} f(i) \\
&= H_N \sum_{i=1}^{N} (-1)^i \binom{N}{i} f(i) - \sum_{i=1}^{N} \frac{(-1)^i}{i} \binom{N}{i} f(i),
\end{aligned}$$

where $H_N = 1 + \frac{1}{2} + \cdots + \frac{1}{N}$ is the N'th harmonic number. Now taking the expectation, we may note, as in solution 1, that $\mathbb{E}[f(i)] = \frac{N+i}{2}$, as each coordinate choice is uniform in $\{i, i+1, \ldots, N\}$. Hence

$$\begin{aligned}
\mathbb{E}(P_f)'(0) &= H_N \sum_{i=1}^{N} (-1)^i \binom{N}{i} \frac{N+i}{2} - \sum_{i=1}^{N} \frac{(-1)^i}{i} \binom{N}{i} \frac{N+i}{2} \\
&= \frac{H_N}{2} \sum_{i=1}^{N} (-1)^i \cdot i \binom{N}{i} + \left(\frac{H_N \cdot N}{2} - \frac{1}{2} \right) \sum_{i=1}^{N} (-1)^i \binom{N}{i} - \frac{N}{2} \sum_{i=1}^{N} \frac{(-1)^i}{i} \binom{N}{i}.
\end{aligned}$$

We have three sums to evaluate. First note that, from the binomial formula

$$(1-x)^N = \sum_{i=0}^{N} (-1)^i \binom{N}{i} x^i.$$

- Plugging in $x = 1$,

$$0 = \sum_{i=0}^{N} (-1)^i \binom{N}{i} x^i \implies \sum_{i=1}^{N} (-1)^i \binom{N}{i} = -1.$$

- Differentiating and plugging in $x = 1$,

$$N \cdot (1-x)^{N-1} = \sum_{i=0}^{N} (-1)^i \cdot i \binom{N}{i} x^{i-1} \implies 0 = \sum_{i=1}^{N} (-1)^i \cdot i \binom{N}{i},$$

  where the $i = 0$ term is trivially 0.

- The final term may be found in a couple of ways. Firstly,

$$\begin{aligned}
\sum_{i=1}^{N} \frac{(-1)^i}{i} \binom{N}{i} &= \sum_{i=1}^{N} \frac{(-1)^i}{i} \binom{N}{i} x^i \bigg|_{x=1} = \sum_{i=1}^{N} \int_0^1 (-1)^i \binom{N}{i} x^{i-1} \, \mathrm{d}x \\
&= \int_0^1 \frac{(1-x)^N - 1}{x} \, \mathrm{d}x = \int_1^0 \frac{u^N - 1}{1-u} (-1) \, \mathrm{d}u \\
&= -1 \int_0^1 (u^{N-1} + \cdots + u + 1) \, \mathrm{d}u = -1 \left(\frac{u^N}{N} + \cdots + \frac{u^2}{2} + \frac{u}{1} \right) \bigg|_{u=0}^{u=1} \\
&= -H_N.
\end{aligned}$$

Alternatively, we may proceed by induction. For $N = 1$ this term may be calculated to give $-1 = -H_1$, and

$$\begin{aligned}\sum_{i=1}^{N} \frac{(-1)^i}{i}\binom{N}{i} &= \sum_{i=1}^{N} \frac{(-1)^i}{i}\left(\binom{N-1}{i} + \binom{N-1}{i-1}\right) \\ &= \sum_{i=1}^{N-1} \frac{(-1)^i}{i}\binom{N-1}{i} + \sum_{i=1}^{N} \frac{(-1)^i}{i}\binom{N-1}{i-1} \\ &= -H_{N-1} + \frac{1}{N}\sum_{i=1}^{N}(-1)^i\binom{N}{i} = -H_N,\end{aligned}$$

using the first sum we calculated.

Substituting all these terms, we find

$$\mathbb{E}(P_f)'(0) = \frac{H_N}{2}(0) + \left(\frac{H_N \cdot N}{2} - \frac{1}{2}\right)(-1) - \frac{N}{2}(H_N) = \frac{1}{2}.$$

Solution 3. *(solution by contestants)*

Let $\mathcal{P}$ be the set of all such polynomials, i.e. $\mathcal{P} = \{P_f \mid f \in \mathcal{F}\}$. Then, define for each $P \in \mathcal{P}$ the polynomial $h(P)$ by

$$h(P)(x) = N + x - P(x).$$

Then I claim that $h(P) \in \mathcal{P}$, i.e. h is an endomorphism on $\mathcal{P}$. Indeed, notice P takes integer values on $\{1, \ldots, N\}$ if and only if $h(P)$ does, P is of degree less than N if and only if $h(P)$ is as well, and subsequently

$$\begin{aligned} P \in \mathcal{P} &\iff i \leq P(i) \leq N \text{ for all } i \in \{1, \ldots, N\} \\ &\iff N - i \geq N - P(i) \geq 0 \text{ for all } i \in \{1, \ldots, N\} \\ &\iff N \geq N + i - P(i) = h(P)(i) \geq i \text{ for all } i \in \{1, \ldots, N\} \\ &\iff h(P) \in \mathcal{P}. \end{aligned}$$

Now I claim that h is a bijection. This is immediate from the fact $h(h(P)) = N+x-(N+x-P) = P$. Since $\mathcal{P}$ is finite, if P is uniform on $\mathcal{P}$, then so is $h(P)$. Therefore, for P uniformly chosen in $\mathcal{P}$,

$$\begin{aligned}\mathbb{E}[P'(0)] &= \frac{1}{2}\left(\mathbb{E}[P'(0)] + \mathbb{E}[h(P)'(0)]\right) = \frac{1}{2}\left(\mathbb{E}[(P + h(P))'(0)]\right) \\ &= \frac{1}{2}\mathbb{E}[(N + x)'(0)] = \frac{1}{2} \cdot 1 = \frac{1}{2},\end{aligned}$$

by linearity of expectation and derivative.

Solution 4. *(solution by contestants)*

Setting this up algebraically, the polynomial P_f which satisfies $P_f(i) = f(i)$ has coefficients $a_0, a_1, \ldots, a_{N-1}$ given by the following equation:

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 4 & \cdots & 2^{N-2} & 2^{N-1} \\ 1 & 3 & 9 & \cdots & 3^{N-2} & 3^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & N-1 & (N-1)^2 & \cdots & (N-1)^{N-2} & (N-1)^{N-1} \\ 1 & N & N^2 & \cdots & N^{N-2} & N^{N-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-2} \\ a_{N-1} \end{pmatrix} = \begin{pmatrix} f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N-1) \\ f(N) \end{pmatrix}.$$

This matrix V on the left is a Vandermonde matrix, which is invertible with inverse V^{-1}. Hence we can analogously write this as

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{pmatrix} = V^{-1} \begin{pmatrix} f(1) \\ f(2) \\ \vdots \\ f(N) \end{pmatrix}.$$

Notice $(P_f)'(0) = a_1$. hence $\mathbb{E}(P_f)'(0) = \mathbb{E}[a_1]$. But the equation on the right is a linear equation in the $f(i)$, hence

$$\begin{pmatrix} \mathbb{E}[a_0] \\ \mathbb{E}[a_1] \\ \vdots \\ \mathbb{E}[a_{N-1}] \end{pmatrix} = \mathbb{E}V^{-1} \begin{pmatrix} f(1) \\ f(2) \\ \vdots \\ f(N) \end{pmatrix} = V^{-1} \begin{pmatrix} \mathbb{E}[f(1)] \\ \mathbb{E}[f(2)] \\ \vdots \\ \mathbb{E}[f(N)] \end{pmatrix} = V^{-1} \begin{pmatrix} (N+1)/2 \\ (N+2)/2 \\ \vdots \\ (N+N)/2 \end{pmatrix}.$$

But since V^{-1} is the inverse of the V, $V^{-1}V = I$, so we get

$$V^{-1} \begin{pmatrix} N/2 \\ N/2 \\ \vdots \\ N/2 \end{pmatrix} = \begin{pmatrix} N/2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad V^{-1} \begin{pmatrix} 1/2 \\ 2/2 \\ \vdots \\ N/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ \vdots \\ 0 \end{pmatrix}.$$

This shows that $\mathbb{E}[a_1] = 1/2$, as desired.

Problem 4.

(proposed by Dylan Toh)

Let $(t_n)_{n\geq 1}$ be the sequence defined recursively by $t_1 = 1$, $t_{2k} = -t_k$, and $t_{2k+1} = t_{k+1}$ for all $k \geq 1$. Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{t_n}{\sqrt[2024]{n}}.$$

(a) Prove that the series converges to a real number c.

(b) Prove that c is non-negative.

(c) Prove that c is strictly positive.

Notes on Marking. 2 marks were awarded for (a), with 1 partial mark for establishing the convergence of a subsequence of partial sums. 4 marks were awarded for (b), with 1 partial mark for applying the mean value theorem or equivalent. 4 marks were awarded for (c). No marks were awarded for establishing properties of (t_n) or attempting common series convergence tests.

Solution 1.

(solution by Dylan Toh)

Note (t_n) is the Thue-Morse sequence: one may show by (strong) induction that

$$t_n = (-1)^{(\text{number of 1's when } n-1 \text{ is written in binary})}.$$

Thus the following properties follow: $t_{n+2^m} = -t_n$ for $1 \leq n \leq 2^m$; and more generally, $t_{n+2^m(k-1)} = t_n t_k$ for $1 \leq n \leq 2^m, k \geq 1$. These properties will be used to bound partial sums of the series, via iterated applications of the mean value theorem (MVT).

We investigate in generality the sum

$$L = L(s) = \sum_{n=1}^{\infty} t_n n^{-s}$$

for $s > 0$; the problem then concerns the value of $c = L(\frac{1}{2024})$.

(a) **Convergence of $L(s)$ for $s > 0$:** Let $L_N = \sum_{n=1}^{N} t_n n^{-s}$ be the partial sums. Note

$$\lim_{N\to\infty} L_{2N} = \sum_{n=1}^{\infty} \left(t_{2n-1}(2n-1)^{-s} + t_{2n}(2n)^{-s}\right) = \sum_{n=1}^{\infty} t_n \left((2n-1)^{-s} - (2n)^{-s}\right)$$

is absolutely convergent, since

$$\left|(2n-1)^{-s} - (2n)^{-s}\right| = \left|\left(\frac{d}{dx}x^{-s}\right)_{x\in(2n-1,2n)}\right| \leq s(2n-1)^{-(1+s)}$$

by MVT, and

$$s\sum_{n=1}^{\infty}(2n-1)^{-(1+s)} \leq s\left(1 + \frac{1}{2}\int_1^{\infty} x^{-(1+s)}dx\right) = s\left(1 + \frac{1}{2s}\right) < +\infty.$$

Thus $L_{2N} \to L$ converges as $N \to \infty$.

Consequently, $L_N = L_{2\lfloor N/2\rfloor} + O(N^{-s}) \to L + 0 = L$ converges as $N \to \infty$.

(b) **Iterated MVT:** Let f be a smooth function, and $x \in \mathbb{R}$. We induct on $m \geq 1$ that

$$f_m(x) = \sum_{n=1}^{2^m} t_n f(x+n) = (-1)^m 2^{m(m-1)/2} f^{(m)}(x+\xi)$$

for some $\xi \in (1, 2^m)$ dependent on x.

Base case $m = 1$: $f_1(x) = f(x+1) - f(x+2) = -f'(x+\xi)$ for some $\xi \in (1,2)$, by MVT.

Induction hypothesis $m > 1$: note

$$f_m(x) = f_{m-1}(x) - f_{m-1}(x + 2^{m-1}) = -2^{m-1} f'_{m-1}(x + \xi_1)$$

for $\xi_1 \in (0, 2^{m-1})$, by MVT. By the induction hypothesis,

$$f'_{m-1}(x+\xi_1) = (-1)^{m-1} 2^{(m-1)(m-2)/2} f^{(m)}(x + \xi_1 + \xi_2)$$

for $\xi_2 \in (1, 2^{m-1})$. The result follows by setting $\xi = \xi_1 + \xi_2 \in (1, 2^m)$.

(c) **Bound on L:** Setting $f(x) = x^{-s}$ and grouping terms into blocks of 2^m terms,

$$\begin{aligned} L = \lim_{N\to\infty} L_{2^m N} &= \sum_{k=1}^{\infty} \sum_{n=1}^{2^m} t_{2^m(k-1)+n} f(2^m(k-1)+n) \\ &= \sum_{k=1}^{\infty} t_k f_m(2^m(k-1)) \\ &= \sum_{k=1}^{\infty} t_k (-1)^m 2^{m(m-1)/2} f^{(m)}(2^m(k-1)+\xi_k), \end{aligned}$$

where $\xi_k \in (1, 2^m)$ for all $k \geq 1$.

Note $f^{(m)}(x) = (-1)^m s(s+1)(s+2)\dots(s+m-1)x^{-(s+m)}$. Thus,

$$L = s(s+1)(s+2)\dots(s+m-1)2^{m(m-1)/2} \sum_{k=1}^{\infty} t_k (2^m(k-1)+\xi_k)^{-(s+m)}.$$

Since $x^{-(s+m)}$ is decreasing, and $t_1 = 1$, thus

$$\begin{aligned} \sum_{k=1}^{\infty} t_k (2^m(k-1)+\xi_k)^{-(s+m)} &\geq (2^m)^{-(s+m)} - (2^m+1)^{-(s+m)} - \sum_{r\geq 2} (2^m r + 1)^{-(s+m)} \\ &\geq (2^m)^{-(s+m)} \left(1 - \left(1 - \frac{1}{2^m+1}\right)^{s+m} - \sum_{r\geq 2} r^{-(s+m)} \right) \\ &\geq (2^m)^{-(s+m)} \left(1 - \left(1 - \frac{1}{2\cdot 2^m}\right)^{m} - \sum_{r\geq 2} r^{-m} \right). \end{aligned}$$

To show the limit $L > 0$ is strictly positive, it suffices to pick an $m \in \mathbb{N}$ such that the expression E enclosed in the brackets is strictly positive. Using the bound $(1-x)^m \geq 1 - mx + \binom{m}{2}x^2$ for $0 < x < 1$ (Bonferroni's/true by induction), we may bound the expression E by

$$\begin{aligned} E &\geq \frac{m}{2} 2^{-m} - \binom{m}{2} 2^{-2m-2} - 2^{-m} - \sum_{r\geq 3} r^{-m} \\ &\geq m 2^{-m} \left(\frac{1}{2} - m 2^{-m} - \frac{1}{m} - \frac{2^m}{m} \int_2^{\infty} x^{-m} dx \right) \\ &= m 2^{-m} \left(\frac{1}{2} - m 2^{-m} - \frac{1}{m} - \frac{1}{m(m-1)} \right) \end{aligned}$$

with $m2^{-m}, \frac{1}{m}, \frac{1}{m(m-1)} \to 0$ as $m \to \infty$. Thus $E > 0$ for sufficiently large m, and $L > 0$. $\square$

Comment. One may directly use iterated MVT to show $L_{2^m} \geq 0$ for all $m \in \mathbb{N}$, thus the limit $L \geq 0$. However, strict positivity $L > 0$ requires a more careful bounding.

Comment. One may work out a concrete positive bound with $m = 2$ (i.e. grouping terms into blocks of 4), if one is careful with explicit computations:

$$s^{-1}L(s)\big|_{s=1/2024} \geq \left. \frac{1 - 2^{-s} - 3^{-s} + 4^{-s}}{s} - 2(s+1)\sum_{r\geq 1}(4r+1)^{-(s+2)} \right|_{s=1/2024}$$

$$\geq 0.35 - 2.001\left(5^{-2} + \sum_{r\geq 9} r^{-2}\right) \geq 0.35 - 2.001\left(5^{-2} + 8^{-1}\right) > 0.$$

Solution 2. *(solution by Timur Pryadilin with minor additions by Anubhab Ghosal)*

Fix $\alpha = \frac{1}{2024}$. Note that $t_n \in \{1, -1\}$ and that

$$t_{4k+1} = -t_{4k+2} = -t_{4k+3} = t_{4k+4} = t_{k+1}.$$

As $\frac{1}{n^\alpha} \to 0$, it suffices to consider the convergence of the series

$$S := \sum_{n \in 4\mathbb{Z}_{\geqslant 0}+1} t_n\left(\frac{1}{n^\alpha} - \frac{1}{(n+1)^\alpha} - \frac{1}{(n+2)^\alpha} + \frac{1}{(n+3)^\alpha}\right) = \sum_{n \in 4\mathbb{Z}_{\geqslant 0}+1} \frac{t_n}{n^\alpha} f\Big(\frac{1}{n}\Big),$$

where $f(x) := 1 - (1+x)^{-\alpha} - (1+2x)^{-\alpha} + (1+3x)^{-\alpha}$.

One computes the successive derivatives of f to get that $f(0) = f'(0) = 0$, $f''(0) = 4\alpha(\alpha+1)$ and that

$$f'''(x) = \alpha(\alpha+1)(\alpha+2)\big((1+x)^{-\alpha-3} + 8(1+2x)^{-\alpha-3} - 27(1+3x)^{-\alpha-3}\big).$$

For $x \in [0, \frac{1}{5}]$, $(1+3x) \leqslant \frac{4}{3}(1+x)$ and $(1+2x) \geqslant (1+x)$ and so

$$f'''(x) \leqslant \alpha(\alpha+1)(\alpha+2)(1+x)^{-\alpha-3}\left(9 - 27\Big(\frac{3}{4}\Big)^{3+\alpha}\right) \leqslant 0 \text{ for } x \in \Big[0, \frac{1}{5}\Big].$$

Letting $h(x) = f(x) - 2\alpha(\alpha+1)x^2$, one has $h(0) = h'(0) = h''(0) = 0$ and that $h'''(x) = f'''(x) \leqslant 0$ for $x \in [1, \frac{1}{5}]$. Therefore,

$$f(x) \leqslant 2\alpha(\alpha+1)x^2 \text{ for } x \in \Big[0, \frac{1}{5}\Big].$$

It follows that S is absolutely convergent and that

$$|S - f(1)| \leqslant 2\alpha(\alpha+1)\sum_{n \in 4\mathbb{N}+1} \frac{1}{n^{2+\alpha}} \leqslant 2\alpha(\alpha+1)\frac{1}{4^2}\zeta(2) = \alpha(\alpha+1)\frac{\pi^2}{48} < \frac{\alpha}{4}.$$

Using the inequalities $1 - x \leqslant e^{-x} \leqslant 1 - x + \frac{x^2}{2}$, one can bound

$$f(1) \geqslant \alpha\left(\log\Big(\frac{3}{2}\Big) - \frac{\alpha}{2}(\log(2)^2 + \log(3)^2)\right) > \frac{\alpha}{4},$$

and we are done.

Problem 5. *(proposed by Dylan Toh)*

Is it possible to dissect an equilateral triangle into 3 congruent polygonal pieces (not necessarily convex), one of which contains the triangle's centre in its interior?

Note: The interior of a polygon does not include its perimeter.

Notes on Marking. 2 marks were awarded for addressing the case where a piece touches two corners of $\triangle$. 4 marks were awarded for formulating the notion of a nice vertex (a 60° vertex shared by its convex hull), and addressing the various cases of 1, 3, or more than 3 nice vertices. In the final case of 2 nice vertices, 1 mark was awarded for showing some two pieces are related by a symmetry of $\triangle$. 1 mark was awarded for resolving the case where the symmetry is a rotation, and 2 marks awarded for resolving the case where the symmetry is a reflection. No marks were awarded for the correct answer, or showing that there is a vertex common to all 3 polygons.

Solution 1. *(solution by Dylan Toh)*

No. Refer to the equilateral triangle as $\triangle$ (WLOG of side length 1) with centre O. Suppose otherwise that one may dissect $\triangle$ into identical polygons $P_1, P_2, P_3 \cong P$, one of which contains O in its interior. We adopt a two-step proof:

1. By considering how the vertices of $\triangle$ are distributed among the pieces P_i, we conclude that two of the pieces are related by either a 120° rotation about O (called a 'central rotation'), or a reflection about a reflection axis of $\triangle$ (called an 'axial reflection').

2. We then derive a contradiction in either case.

Let $\tilde{P}$ and $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3$ denote the convex hulls of the polygons. We call a vertex of P 'nice' it has internal angle 60°, and also corresponds to a vertex of $\tilde{P}$ of the same internal angle 60°.

- Case I: a piece touches two corners of $\triangle$. Then P has two vertices of distance 1 apart. Since the only pairs of points in $\triangle$ of distance 1 apart are a pair of vertices, so each piece P_i must touch (at least) two corners of $\triangle$.

  Therefore, each P_i contains a path in its interior (except for its endpoints) between two corners of $\triangle$. Drawing these paths out, at least one path (in P_1, say) bounds a region with a side of $\triangle$ not containing the other two paths; consequently, P_1 must share a full side with a side of $\triangle$. Since the pieces are congruent, thus each piece P_i shares a full side with $\triangle$ as well.

  Thus P has a unique side of length 1, called the 'long side'; fixing an orientation of the boundary of P, the position/orientation of a copy of P is thus determined by the position/orientation of that side. By pigeonhole, two of the long sides of P_1, P_2, P_3 are oriented in the same direction around $\triangle$ (say P_1, P_2); they are thus related by a central rotation (see Case A below).

- Case II: Each of the 3 corners of $\triangle$ is a vertex of a unique P_i. This corresponds to a nice vertex of each P_i.

  Note $\tilde{P}$ has ≤ 3 vertices of internal angle 60° (since the total sum of external angles of convex polygon $\tilde{P}$ is 360°, but each nice vertex contributes an external angle of 120°), with equality if and only if $\tilde{P}$ is equilateral. If $\tilde{P}$ were equilateral, then it must have side length $s > \frac{1}{\sqrt{3}} > \frac{1}{2}$ for a piece P_i to contain the centre. But this means (identifying a side of $\triangle$ with interval $[0,1]$) there is a path from 0 to s inside some P_1, and a path from $1-s$ to 1 inside some other P_2; these paths must intersect, implying that the interiors of P_1 and P_2 intersect, a contradiction.

Thus $\tilde{P}$ has ≤ 2 vertices of internal angle $60°$, so P has ≤ 2 nice vertices. Meanwhile, each corner of P_i agreeing with a corner of $\triangle$ corresponds to a nice vertex of P. By pigeonhole, two of them must agree (say, corresponding to P_1, P_2). The two possible cases of orientation then show that P_1, P_2 are either related by a central rotation (see Case A below) or an axial reflection (see Case B below).

In both cases above, one concludes that there are two pieces P_1, P_2 related either by a central rotation or an axial reflection.

- Case A: P_1, P_2 are related by a central rotation. Let τ be this rotation; one has $\tau P_1 = P_2$. Let $\hat{P}_3 = \tau P_2 = \tau^{-1} P_1$; note that $\hat{P}_3 \cap P_1 = \tau^{-1}(P_1 \cap P_2)$ and $\hat{P}_3 \cap P_2 = \tau(P_2 \cap P_1)$, so $\hat{P}_3$ has disjoint interiors with P_1, P_2 (but is congruent). We must thus have equality $P_3 = \hat{P}_3$ (e.g. by an area argument: P_3 must contain $\hat{P}_3$, but they are congruent), and all pieces are related by rotation about O; thus all three pieces contain O, a contradiction.

- Case B: P_1, P_2 are related by an axial reflection about axis l. Since this axis passes through O, thus O is not in the interior of P_1 or P_2 (since it can't be in both interiors); it is thus contained in the interior of P_3, which is also reflectionally symmetric about l.

  If P only had one nice vertex, then (by the argument in Case II above) the pair P_1, P_3 is also related by either a central rotation (in which Case A derives a contradiction) or an axial reflection (which would imply O is also not in the interior of P_3, a contradiction).

  Thus P has 2 nice vertices, and (by the reflectional symmetry of P_3 about l) the other nice vertex of P_3 must lie on l as well. But then O lies on the line segment between the two nice vertices of P_3 (which is also the angle bisector of either nice vertex, and has length $> \frac{1}{\sqrt{3}}$). This line segment also lies on l, and is thus contained in the interior of P_3, since P_1, P_2 are reflectionally symmetric about l. We may furthermore assume that we are in Case II above (since Case I is resolved by Case A). Thus P_1, P_2 also have nice vertices agreeing with corners of $\triangle$, thus they contain the respective angle bisectors of length $> \frac{1}{\sqrt{3}}$. So all three pieces contain O, a contradiction. $\square$

Comment. It is not known whether a circle can be dissected into finitely many identical pieces, one of which contains the centre in its interior. One may wish to investigate if a dissection of the equilateral triangle into 3 similar polygons (i.e. identical up to scaling) is possible.