IMPERIAL-CAMBRIDGE
MATHEMATICS
COMPETITION

ICMC 8 — Round One

24 November 2024

Name: ______________________________

Contestant ID: ______________________________

University: ______________________________

Instructions:

- Do not turn over until told to do so.
- You will have 3 hours to solve 6 problems, each of which carries 10 marks.
- Use a black or blue pen or a dark pencil. Rulers, compasses, protractors, and erasers may be used but will not be required. All electronic devices, including calculators, are prohibited.
- Drinks are allowed, but food is prohibited.
- Write your solution to each problem on a different page. At the top of each page, write down the question number, your initials, and your contestant number. Use both sides whenever possible. Writing clearly and not too faintly – your work will be scanned for marking.
- Problems are listed roughly in order of difficulty. Proofs are expected for all problems even if they only ask for an answer.
- One complete solution will be awarded more marks than several unfinished attempts.
- You may not leave the contest venue early unless exceptional circumstances arise.
- Do not take away the problems sheet or any rough work when leaving the venue.

Declaration:

I hereby declare that the work I am submitting is entirely my own, and that I will not discuss or disclose the contents of this paper until 14:00, 24 November 2024, UTC Time.

Signature: ______________________________

Problem 1. Joe the Jaguar is on an infinite grid of unit squares, starting at the centre of one of them. At the k-th minute, Joe must jump a distance of k units in any direction.

For which n is it possible for Joe to be inside or on the edge of the starting square after n minutes?

Problem 2. Alice and the Mad Hatter are playing a game. At the start of the game, three 2024's are written on the blackboard. Then, Alice and the Mad Hatter alternate turns, with the Mad Hatter starting. On the Mad Hatter's turn, he must pick one of the numbers on the blackboard and increase it by 1. On Alice's turn, she must:

- pick one of the numbers on the blackboard and decrease it by 1, and then
- replace the two numbers a and b on the blackboard which were not chosen by the Mad Hatter on the previous turn with $\sqrt{ab}$.

Alice wins if, on the start of her turn, any of the three numbers are less than 1.

Can the Mad Hatter prevent Alice from winning?

Problem 3. Let V be a subspace of the vector space $\mathbb{R}^{2\times 2}$ of 2-by-2 real matrices. We call V *nice* if for any linearly independent $A, B \in V$, $AB \neq BA$. Find the maximum dimension of a nice subspace of $\mathbb{R}^{2\times 2}$.

Problem 4. Let a *chain* denote a row of positive integers which continue infinitely in both directions, such that for each number n, the n numbers directly to the left of n yield n distinct remainders upon division by n.

(a) If a chain has a maximum integer, what are the possible values of that integer?

(b) Does there exist a chain which does not have a maximum integer?

Problem 5. A positive integer is a *non-trivial perfect power* if it can be expressed as n^k where n and k are positive integers and $k > 1$. Show that there exist arbitrarily large consecutive square numbers with no other non-trivial perfect powers between them.

Problem 6. A set of points in the plane is called *rigid* if each point is equidistant from the three (or more) points nearest to it.

(a) Does there exist a rigid set of 9 points?

(b) Does there exist a rigid set of 11 points?