IMPERIAL-CAMBRIDGE
MATHEMATICS
COMPETITION

8th Edition (2024–2025)

ROUND ONE

Official Solutions*

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*A solution may receive full or partial marks even if it does not appear in this booklet.

Problem 1. *(proposed by Joe Devine)*

Joe the Jaguar is on an infinite grid of unit squares, starting at the centre of one of them. At the k-th minute, Joe must jump a distance of k units in any direction.

For which n is it possible for Joe to be inside or on the edge of the starting square after n minutes?

Notes on Marking. Most contestants performed well on this problem. The most common approach was to reduce the problem to a small number of base values for n with triangle inequality-style arguments, from which a repeating algorithm allows staying inside the square for any n greater than the base values. 2 marks were awarded for discussing the impossibility of cases $n = 1$ and $n = 2$. 1 mark was deducted from this if the $n = 2$ case was simply stated as being obvious, without any arguments. 5 marks could be earned by finding an algorithm that follows from the base cases (e.g. displacement of 0 in four moves in Solution 1) and also the relevant base cases. If some base cases implied by the contestant's algorithm were not discussed, marks were subtracted accordingly. 1 mark was awarded for invoking the triangle inequality. 2 final marks were awarded for showing that the implied base values of n (jumps) can be used to stay within the starting square.

The other main approach was to use arguments similar to those in Solutions 3 and 4. For the former, full marks were awarded if an argument was provided as to why the quadratic inequality holds for $n > 3$. For the latter, full marks were awarded if the inductive proof was fully argued. Both solutions also required discussion of the $n = 1$ and $n = 2$ cases for full marks, similarly to the above scheme.

About one in ten contestants misunderstood the question by either assuming that the jumps have to be parallel to the axes or assuming that "on the edge of" means on adjacent squares. In these cases partial marks were given if the arguments used were relevant to solving the problem without making the incorrect assumptions.

Solution 1. *(solution by Gergely Rozgonyi)*

We will show that it is possible for all $n \geq 3$. First, we note that the order of the jumps for a given n does not affect the outcome as vector addition is communitative.

- Case $n = 1$: After the jump Joe ends up on the unit circle which overlaps the entirety of the starting square, as the furthest point of the starting square is at a distance $\sqrt{2}/2 < 1$ distance from the origin.

- Case $n = 2$: Since the unit circle has diameter 2, the locus of points of distance 2 from any point on the circumference of the unit circle is entirely outside or on the unit circle, so it ends up being a distance of at least 1 away from the origin. Hence, Joe cannot arrive back to the starting square in two jumps either.

- Case $n \geq 3$: Note that for any 4 consecutive positive integers $\{n, n+1, n+2, n+3\}$, Joe can group them into the two sets $\{n, n+3\}$ and $\{n+1, n+2\}$ whose sums are equal. Therefore, by working backwards from n, Joe can assign each group of 4 consecutive jumps to two sets so that the sum of the two sets remains the same. Joe continues this until he is left with no jumps (when $n = 4k$); the unit jump (when $n = 4k+1$); the jumps of length 1 and 2 (when $n = 4k+2$); or the jumps of length 1, 2 and 3 (when $n = 4k+3$).

When $n = 4k$, Joe can jump to the right for all numbers in the first set and jump to the left for all numbers in the second set to end up back at the origin. When $n = 4k + 3$, Joe can first assign 1 and 2 to one of the sets and 3 to the other, and simliarly end up back at the origin.

For the other two cases, since $n \geq 5$, the elements in each of the constructed sets will have a sum L that is greater than $5 + 2 = 7$. Hence, Joe can construct an isosceles triangle with base equal to 1 (for $n = 4k + 1$) or $1 + 2 = 3$ (for $n = 4k + 2$), and legs of length L. Orienting and position this triangle so that one of its vertices is the origin, Joe can follow this triangle to end up back at the origin after finishing his jumps. □

Solution 2.

(solution by Tony Wang)

The $n = 1$ and $n = 2$ cases are impossible (see Solution 1). For $n \geq 3$ we consider the following two cases:

- Case when n is odd: Joe can construct an isosceles triangle with base length $\sqrt{2}/2$ and legs of length $1 + 2 = 3$ each, using the first three jumps to achieve a displacement equal to $\sqrt{2}/2$ and jump to the upper right corner of the square.

  If $n > 3$, then for each pair of consecutive jumps after 3, Joe can ensure that he achieves a displacement of precisely 1 along the line joining the top-right and bottom-left corners of the square: $+k - (k + 1) = -1$ and $-k + (k + 1) = +1$. Therefore he can alternate jumping towards the top-right and bottom-left corners every two minutes, staying inside the starting square.

- Case when n is even: Joe can construct an isosceles triangle with base length $\sqrt{2}/2$ and legs of length $1 + 4 = 2 + 3 = 5$ each, using the first four jumps to achieve a displacement equal to $\sqrt{2}/2$ and jump to the upper right corner of the square.

  If $n > 4$, then we make an argument similar to the one in the case when n is odd to finish. □

Solution 3.

(solution by contestants)

The $n = 1$ and $n = 2$ cases are impossible (see Solution 1). For $n = 3$, Joe can jump $1 + 2 = 3$ to the right, then 3 to the left to reach the origin.

To be able to construct an n-gon with the jumps corresponding to its sides, it is a necessary and sufficient condition that the longest side is shorter then the sum of all other side lengths:

$$\sum_{i=1}^{n-1} i = \frac{(n-1)n}{2} > n,$$

which implies $n < 0$ or $n > 3$. Therefore Joe can construct an n-gon with one of its vertices being the origin for $n \geq 4$, guaranteeing he returns to the origin. □

Solution 4.

(solution by contestants)

This solution is a constructive equivalent of Solution 3. The $n = 1$ and $n = 2$ cases are impossible (see Solution 1). For $n \geq 3$ we proceed by induction.

- Base case $n = 3$: Joe can jump $1 + 2$ to the right, so at time 2 he is at distance 3 from the origin. Then at time 3 he can jump back to the origin.

- Inductive hypothesis: Suppose that Joe can arrive back at the origin in $n = k - 1$ jumps.
- Induction step $n = k$: Denote the circle centred at the origin of radius r by C_r. By the hypothesis Joe can reach C_{k-1} in $k - 2$ jumps. The goal is to use the k^{th} jump to jump to the origin, meaning that at time $k - 1$ Joe must jump from C_{k-1} to C_k.

  This is equivalent to constructing a triangle of side lengths $k - 1$, $k - 1$, and k, which is clearly possible for $k - 1 \geq 3$. Therefore Joe can reach the origin in $n = k$ jumps, and the hypothesis holds $\forall n \geq 3$. $\square$

Problem 2. *(proposed by Tony Wang)*

Alice and the Mad Hatter are playing a game. At the start of the game, three 2024's are written on the blackboard. Then, Alice and the Mad Hatter alternate turns, with the Mad Hatter starting. On the Mad Hatter's turn, he must pick one of the numbers on the blackboard and increase it by 1. On Alice's turn, she must:

- pick one of the numbers on the blackboard and decrease it by 1, and then
- replace the two numbers a and b on the blackboard which were not chosen by the Mad Hatter on the previous turn with $\sqrt{ab}$.

Alice wins if, on the start of her turn, any of the three numbers are less than 1.

Can the Mad Hatter prevent Alice from winning?

Notes on Marking. Unfortunately some contestants misunderstood the question, mostly by thinking Alice and the Mad Hatter pick 1 of the 12 digits written on the board, or by misunderstanding how a and b are chosen. (To be clear, they are the numbers that the Mad Hatter did not choose, meaning there is only one possible pair of numbers a and b can be.) In these cases, we still awarded marks if any of the ideas presented were applicable to the correct version of the problem. 2 marks were awarded for considering the sum of the numbers, 5 marks were awarded for proving that the sum of the three numbers is strictly decreasing, with a 1 mark deduction if strictness is not achieved. On top of this, stating that Alice should choose the smaller of the two numbers the Mad Hatter didn't choose was worth an extra mark. On bounding the decrease, we felt that it was necessary to acknowledge that $\sqrt{x+1} - \sqrt{x}$ (or similar) decreased as x increased and that x was bounded above, so marks were deducted for vagueness.

Solution 1. *(solution by Tony Wang)*

The answer is no. To prove this, we will demonstrate that the sum of the numbers on the blackboard always decreases by at least some $\varepsilon > 0$ after each of Alice's turns. This is sufficient since after more than $6071/\varepsilon$ rounds, if Alice has not won yet, the sum of the numbers will be less than 1, and hence some number must be less than 1 after the Mad Hatter's turn.

Note that the sum never increases after a round, since the Mad Hatter's $+1$ and Alice's -1 cancel to leave the sum unchanged, while the replacement step can only decrease the sum since $a + b \geq 2\sqrt{ab}$ by AM-GM. This means that the sum of the numbers, and by extension the individual numbers, never exceeds 6072 just before the second part of Alice's turn. More precisely, by replacing a and b with $\sqrt{ab}$, Alice decreases the sum by

$$a + b - 2\sqrt{ab} = (\sqrt{a} - \sqrt{b})^2. \qquad (1)$$

If Alice always chooses to decrease the smaller of the two numbers not chosen by the Mad Hatter on the previous turn, then a and b will differ by at least 1, while they are each at most 6072.

WLOG, assume that $a > b$. Now note that (a) $\sqrt{a} - \sqrt{b}$ is minimised for some fixed a when b is as close to a as possible, and (b) it is minimised for some fixed $a - b$ when both a and b are as large as possible. This implies that the minimum value of $\sqrt{a} - \sqrt{b}$ given that $a - b \geq 1$ and $a, b \leq 6072$ is $\sqrt{6072} - \sqrt{6071}$. Hence, we can set $\varepsilon = (\sqrt{6072} - \sqrt{6071})^2$ to finish. $\square$

Solution 2. *(solution by Xu Chen Tan and Freddie Weir [Oxford])*

We follow the previous solution to equation (1), from which we note that

$$\begin{aligned}\left(\sqrt{a}-\sqrt{b}\right)^2 &= \frac{(a-b)^2}{(\sqrt{a}+\sqrt{b})^2} \\ &\geq \frac{1}{(2\sqrt{a})^2} \\ &\geq \frac{1}{4(2024+n)}\end{aligned}$$

since any number on the blackboard must have a value of at most $2024+n$ after the n-th turn. Now, since the harmonic series diverges to infinity, $\sum_{i=1}^{\infty} 1/4(2024+n)$ must also diverge to infinity, and hence the sum of the three numbers must eventually be arbitrarily small. $\square$

Solution 3. *(solution by Brian Reinhart [Oxford])*

Note that on the first move, and after each of Alice's moves, the Mad Hatter will be presented with three numbers on the blackboard, at least two of which will be the same. We will let these numbers be a, a, and b (where a and b are not necessarily distinct). Note that the Mad Hatter has two possible moves:

- Type 1: If the Mad Hatter chooses to increase an a, then Alice can choose to decrease the other a. In this way, the blackboard will have $a+1$, $\sqrt{(a-1)b}$, and $\sqrt{(a-1)b}$. We note as in the above solutions that the sum of the three numbers cannot have increased, while the product decreases by $a^2b-(a+1)(a-1)b=b\geq 1$.

- Type 2: If the Mad Hatter chooses to increase a b, then Alice can choose to decrease an $a-1$. In this way, the blackboard will have $b+1$, $\sqrt{a(a-1)}$, and $\sqrt{a(a-1)}$. We note as in the above solutions that the sum of the three numbers decreases by at least $\varepsilon := (\sqrt{6072}-\sqrt{6071})^2$.

Now, note that if the Mad Hatter plays only finitely many Type 2 moves, then after the last Type 2 move the Mad Hatter will play only Type 1 moves, and eventually the product of the three numbers on the blackboard will decrease until Alice wins. On the other hand, if the Mad Hatter plays infinitely many Type 2 moves, then the sum will decrease until Alice wins. $\square$

Problem 3. *(proposed by Fredy Yip)*

Let V be a subspace of the vector space $\mathbb{R}^{2\times 2}$ of 2-by-2 real matrices. We call V *nice* if for any linearly independent $A, B \in V$, $AB \neq BA$. Find the maximum dimension of a nice subspace of $\mathbb{R}^{2\times 2}$.

Notes on Marking. A proof that the maximum dimension of a nice subspace was strictly less than 4 was worth 2 marks, and a construction for a three-dimensional nice subspace was worth 8 marks. 2 marks could be scored by considering two general matrices of a chosen three dimensional subspace and finding a system of simultaneous equations resulting from these matrices commuting. Two more marks were awarded if it was recognised that these equations correspond to a cross product in $\mathbb{R}^3$, as in Solution 2. 2 marks were deducted if the candidate neglected to consider if certain terms are zero as is done explicitly in Solution 1. No marks were awarded for attempting to construct a two-dimensional nice subspace.

Solution 1. *(solution by Fredy Yip)*

We will show that the answer is 3. Firstly, note that a nice subspace cannot be the entire space, since otherwise it contains both the matrices $A = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$, where indeed A and B are linearly independent but $AB = BA = 0$. Hence, the dimension is at most 3.

Now, we will prove that the subspace generated by $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$ is nice. Let $A = \left(\begin{smallmatrix}a&b\\c&0\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix}d&e\\f&0\end{smallmatrix}\right)$, and suppose that $AB = BA$. Then since

$$AB = \begin{pmatrix} ad+bf & ae \\ cd & ce \end{pmatrix} \quad \text{and} \quad BA = \begin{pmatrix} ad+ec & bd \\ af & bf \end{pmatrix}, \tag{2}$$

we must have $ae = bd$ and $cd = af$. Now, assuming that $d \neq 0$, we have $\lambda e = b$ and $\lambda f = c$ where $\lambda = a/d$, and hence $\lambda B = A$. Otherwise, if $d = 0$, then either $a \neq 0$, in which case $f = e = 0$, or $a = 0$, in which case by looking at (2) we get $ce = bf$. In all cases, we have that A and B are in fact not linearly independent. $\square$

Solution 2. *(solution by contestants)*

This is similar to the previous solution. We again show that the subspace V constructed in that solution is nice. We let $A = \left(\begin{smallmatrix}a&b\\c&0\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix}d&e\\f&0\end{smallmatrix}\right)$, and suppose that $AB = BA$. Then since

$$AB = \begin{pmatrix} ad+bf & ae \\ cd & ce \end{pmatrix} \quad \text{and} \quad BA = \begin{pmatrix} ad+ec & bd \\ af & bf \end{pmatrix},$$

we get the three equations

$$\begin{aligned} bf - ec &= 0 \\ ae - bd &= 0 \\ cd - af &= 0 \end{aligned}$$

which means that, if we view A, B as elements of $\mathbb{R}^3$ in the usual way, we get $A \times B = 0$. Then expanding the formula for the cross product $|A \times B| = |A||B||\sin\theta| = 0$, where θ is the angle between A, B in $\mathbb{R}^3$. Now since we are assuming that neither matrix is zero, we must have $\sin\theta = 0$ meaning that A, B as vectors are collinear. Thus as matrices they are linearly dependent.

Alternatively, notice that the above equations imply that the matrix

$$\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$$

has rank less than or equal to 1. And hence the matrices are linearly dependent. $\square$

Solution 3. *(solution by contestants)*

We will show that the answer is 3. Firstly, note that a nice subspace cannot be the entire space, since otherwise it contains both the matrices $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, where indeed A and B are linearly independent but $AB = BA = 0$. Hence, the dimension is at most 3.

Now, we will prove that the subspace generated by $A \coloneqq \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $B \coloneqq \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$, and $C \coloneqq \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ is nice. Note that

$$AB - BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad BC - CB = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad AC - CA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are all linearly independent. Then let two general elements of V be $X = aA + bB + cC$ and $Y = a'A + b'B + c'C$. If X and Y commute, then

$$XY - YX = (ab' - a'b)(AB - BA) + (bc' - c'b)(BC - CB) + (ac' - ca')(AC - CA) = 0.$$

However, we know that the matrices in the above expression are linearly independent and hence we find that $ab' - a'b = bc' - c'b = ac' - ca' = 0$. In other words, the matrices

$$\begin{pmatrix} a & a' \\ b & b' \end{pmatrix}, \quad \begin{pmatrix} b & b' \\ c & c' \end{pmatrix}, \quad \begin{pmatrix} a & a' \\ c & c' \end{pmatrix},$$

are all singular. Hence we can find real numbers λ and μ such that $\lambda a = \mu a'$ and $\lambda b = \mu b'$, and this in turn implies that $\lambda c = \mu c'$ (unless $a = a' = b = b' = 0$, but in this case X, Y are clearly linearly dependent). But then X, Y must be linearly dependent. So V is nice. $\square$

Solution 4. *(solution by contestants)*

This is variation on Solution 3. Once again, V cannot be four dimensional or it contains the identity. Suppose that A, B, C are three linearly independent matrices such that no two commute. We will show that their span is nice. Let

$$D_1 = \lambda_1 A + \lambda_2 B + \lambda_3 C$$
$$D_2 = \mu_1 A + \mu_2 B + \mu_3 C$$

be two general elements of V. Assume that they are linearly independent. Then the vectors

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}, \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

are linearly independent. This is equivalent to each of

$$\det \begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix}, \det \begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_3 & \mu_3 \end{pmatrix}, \det \begin{pmatrix} \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{pmatrix}$$

being non zero. Now we compute

$$
\begin{aligned}
D_1D_2 - D_2D_1 &= (\lambda_1\mu_2 - \lambda_2\mu_1)AB + (\lambda_2\mu_1 - \mu_1\lambda_2)BA \\
&\quad + (\lambda_1\mu_3 - \lambda_3\mu_1)AC + (\lambda_3\mu_1 - \lambda_1\mu_3)CA \\
&\quad + (\lambda_2\mu_3 - \lambda_3\mu_2)BC + (\lambda_3\mu_2 - \lambda_2\mu_3)CB \\
&= \det\begin{pmatrix}\lambda_1 & \mu_1 \\ \lambda_2 & \mu_2\end{pmatrix}(AB - BA) + \det\begin{pmatrix}\lambda_1 & \mu_1 \\ \lambda_3 & \mu_3\end{pmatrix}(AC - CA) \\
&\quad + \det\begin{pmatrix}\lambda_2 & \mu_2 \\ \lambda_3 & \mu_3\end{pmatrix}(BC - CB).
\end{aligned}
$$

Now $AB - BA, AC - CA, BC - CB$ are all linearly independent so if this expression is zero, all of the coefficients are. But this is impossible by our previous remarks. Thus D_1, D_2 do not commute and V will be nice. We just need to construct an example. The matrices

$$A = \begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix}, B = \begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}, C = \begin{pmatrix}0 & 0 \\ 1 & 1\end{pmatrix}$$

can easily be verified to work. $\square$

Solution 5.

(solution by contestants)

This is another proof that the subspace V in Solution 2 is nice. Again if we have $A = \left(\begin{smallmatrix}a & b \\ c & 0\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix}d & e \\ f & 0\end{smallmatrix}\right)$, suppose that they are non zero and linearly independent but $AB = BA$. Then since

$$AB = \begin{pmatrix}ad + bf & ae \\ cd & ce\end{pmatrix} \qquad \text{and} \qquad BA = \begin{pmatrix}ad + ec & bd \\ af & bf\end{pmatrix},$$

we get

$$dA = \begin{pmatrix}ad & bd \\ cd & 0\end{pmatrix} = \begin{pmatrix}ad & ae \\ af & 0\end{pmatrix} = aB$$

but A, B are linearly independent so $a = d = 0$. Then

$$A = \begin{pmatrix}0 & b \\ c & 0\end{pmatrix}, B = \begin{pmatrix}0 & e \\ f & 0\end{pmatrix}.$$

But then

$$eA = \begin{pmatrix}0 & eb \\ ec & 0\end{pmatrix} = \begin{pmatrix}0 & eb \\ fb & 0\end{pmatrix} = bB.$$

Again A, B are linearly independent so $e = b = 0$. But then

$$A = \begin{pmatrix}0 & 0 \\ c & 0\end{pmatrix}, B = \begin{pmatrix}0 & 0 \\ f & 0\end{pmatrix}$$

are linearly dependent (they are non zero by assumption). Thus we have proven by contradiction that A, B cannot be both commutative and linearly independent. V is nice. $\square$

Problem 4. *(proposed by Tony Wang)*

Let a *chain* denote a row of positive integers which continue infinitely in both directions, such that for each number n, the n numbers directly to the left of n yield n distinct remainders upon division by n.

(a) If a chain has a maximum integer, what are the possible values of that integer?

(b) Does there exist a chain which does not have a maximum integer?

Notes on Marking. Part (a) of this problem was harder than intended, whereas part (b) was easier. Most people found that 1 and 2 worked in part (a), although some neglected to give examples or forgot that these values worked entirely. Some tried to prove the $n \geq 3$ case in part (a) by either assuming that consecutive n's were n apart, or by looking at densities, but for example the solution ... 1 2 2 1 2 2 1 2 ... shows that 2 may have density 1/2 or 2/3. Most solutions to part (b) found the solution ... 5 4 3 2 1 1 1 ..., but there were a few wackier solutions. Common mistakes were not reading that the integers were positive, putting down a 0 somewhere, and assuming consecutive k's were k apart, or assuming the chain was cyclic.

Solution 1. *(solution by Ishan Nath)*

We will prove that the answer to part (a) is 1 and 2, and the answer to part (b) is yes.

(a) To show that these work, consider the chains

$$\ldots\ 1\ 1\ 1\ 1\ 1\ 1\ \ldots \qquad \text{and} \qquad \ldots\ 2\ 1\ 2\ 1\ 2\ 1\ \ldots.$$

Hence, it remains to prove that the maximum integer cannot be greater than 2.

To show this, let the maximum integer be $n > 2$ and consider all the occurrences of $n-1$ to the left of some n at position p. The $n-1$ numbers to the left of each occurrence must contain another $n-1$, since it is the only number less than or equal to n that is congruent to $n-1$ mod n, so there must be infinitely many occurrences of $n-1$. A similar argument can be made for all the occurrences of n to the left of position p.

However, we will now prove that each occurrence of $n-1$ and n to the left of position p must be interleaved. Suppose to the contrary that there are two or more occurrences of $n-1$ between some two consecutive n's. Then since the left n must be within a distance of n to the right n, then so too must the occurrences of $(n-1)$'s, but this is a contradiction. A similar argument shows that there cannot be two or more n's between any two consecutive $(n-1)$'s, and so we can conclude that our chain must look like

$$\ldots(n-1)\ldots\ldots n\ldots\ldots(n-1)\ldots\ldots n\ldots\ldots(n-1)\ldots.$$

But now, notice that every $n-1$ to the left of position p must have an n within the $n-1$ numbers to the left of it, and so it cannot also have a 1 in that range. However, this implies that there cannot be any 1's to the left of position p. Since each n requires a 1 to the left of it, this presents a contradiction.

(b) We note that the sequence clearly works:

$$\ldots\ 6\ 5\ 4\ 3\ 2\ 1\ 1\ 1\ 1\ \ldots. \qquad \square$$

Solution 2.

(solution by Tony Wang)

We will use the same constructions to part (a) as in Solution 1.

(a) To show that the maximum integer cannot be greater than 2, we first consider the case of an even maximum integer $n > 2$. Let $n = 2k - 2$ for some positive integer $k \geq 3$, and suppose that the maximum integer of the chain is n. Note that each time n occurs in the chain, the n integers to the left of n must be a permutation of $\{1, 2, \dots, n\}$. In particular, it can only contain a single k. Also, note that each time k occurs in the chain, k must occur again exactly once in the k numbers directly to the left, since k is the only number congruent to 0 modulo k less than n.

The two facts above imply that, for some n in the chain (that is far-enough left of the rightmost occurrence of n, if one exists), the position of the k in the permutation of $\{1, 2, \dots, n\}$ left of the n cannot be too far to the right (for that would necessitate another k to appear within the "permutation"), nor can it be too far to the left (for then this k would not be "in range" of the the next k to the right, which itself must be to the right of the n). In fact, it turns out that the k must be in position k of the permutation, resulting in a pattern which looks like this:

$$\dots n\ \overbrace{\underbrace{k\,\dots\dots\dots\, n}_{k}\ \underbrace{k\,\dots\dots\dots\, n}_{k}}^{n=2k-2}\ k\dots.$$

This implies that, far-left enough, each k must have an n directly to the left of it. By a similar argument analysing the relationship between k and $n-1$, we see that each k must have an $n-1$ within the two numbers left of it. Since n already occupies the position directly to the left, that means $n-1$ must occupy the position two spots to the left.

However, we now have an $n-1$ directly to the left of an n. This is a contradiction, as the $n-1$ necessitates another $n-1$ within the $n-1$ numbers to the left of it, while the n cannot accommodate another $n-1$ in the same range.

If n is odd instead, then we let $n = 2k-1$ for $k \geq 2$, and we will arrive at a contradiction when we try to find a suitable position to place k:

$$\dots n\ \overbrace{\underbrace{k\,\dots\dots\dots}_{k}\ \underbrace{k\,\dots\dots\dots\, n}_{k}}^{n=2k-1}\ k\dots.$$

(b) We note that recursively taking the factorial of the previous highest number as we go to the left works:

$$\dots\ 719\ \ 720\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 1\ \ 2\ \ 3\ \ 1\ \ 1\ \ 1\ \ 1\ \ \dots.$$ $\square$

Solution 3.

(solution by Daniel Naylor [Cambridge])

Let us say that x is *in range of* y, or that y *includes* x, if x is in the y numbers to the left of y.

For part (a), suppose for the sake of contradiction that $n > 2$ is the maximum integer in the chain, and suppose there is an n at position p. Since each n needs a permutation of $\{1, 2, \dots, n\}$ in the n positions to the left of itself, we will have infinite many copies of each positive integer less than or equal to n to the left of position p.

Consider an arbitrarily long sequence of consecutive 1's. Note that there cannot be two or more occurrences of 1 between two consecutive n's, since the left n must be in range of the right n, but that would imply both 1's are in range as well, a contradiction. Hence, to the right of each 1, there must be an occurrence of n before there is another occurrence of 1. Consider the sequence of n's generated by taking each 1 and finding the nearest n to the right of it. Note that this yields an arbitrarily long sequence of n's, although this sequence is not necessarily a consecutive sequence of n's. We now have an alternating sequence of n's and 1's, where there are no other n's between each 1 and its corresponding n to the right:

$$\dots 1\underbrace{\dots\dots}_{n\text{-free}}n\underbrace{\dots\dots}_{a}1\underbrace{\dots\dots}_{n\text{-free}}n\underbrace{\dots\dots}_{b}1\underbrace{\dots\dots}_{n\text{-free}}n\dots.$$

Note that there also cannot be an $n-1$ in the n-free regions, since the range of such an $n-1$ would go further left than the range of the n to the right of it, which means that it would include both the n and the 1 that the n to the right of it includes, a contradiction. Now suppose that there is no $n-1$ in the range with length b – in this case the closest an $n-1$ can be to the left of the rightmost n is in the range with length a, but this implies that the two rightmost 1's must be in the range of the rightmost n, a contradiction. Hence we have this sequence of numbers:

$$\dots\underbrace{\overbrace{n\dots\dots1}^{a}\dots\dots}_{\le n}\underbrace{n\dots\dots(n-1)\dots\dots1}_{b}\dots$$

But now, letting $A =$ number of numbers between the leftmost n and the $n-1$ (inclusive), and $B =$ number of numbers between the leftmost n and the rightmost 1 (inclusive), we have that

$$a+n-1+1 \le A < B \le n+b \qquad \Longrightarrow \qquad a<b.$$

But this means that the distance between each of the $n\dots1$ pairs is strictly decreasing as we go to the left, a contradiction by infinite descent. $\square$

Solution 4. *(solution by Massimiliano Foschi [SNS], Xu Chen Tan, and Tony Wang)*

For part (a), suppose for the sake of contradiction that $n>2$ is the maximum integer in the chain. Let $k=\lfloor n/2\rfloor+1$, so that k is the smallest integer such that $2k>n$. Now consider all the occurrences of k to the left of some n at position p, and suppose that the rightmost such k is at position p_k. The k numbers to the left of each k must contain some number which is 0 mod k, meaning that there must be at least one k in every contiguous block of k numbers to the left of position p_k. Hence, the limit inferior of the density of k's in any contiguous block of numbers that starts at position p_k (as the rightmost number) and includes $m\to\infty$ numbers in total must be least $1/k$.

But now we note that, by a similar argument to the ones presented in Solutions 1 and 2, for each a such that $k<a\le n$, we must have an a in the gap between any two consecutive k's left of p_k, and so the limit inferior of the density of a's as above must also be at least $1/k$.

Meanwhile, for any a such that $a<k$, note that, left of position p, we cannot have more than two occurrences of n between any two a's. To prove this, suppose to the contrary that there are three or more n's between two consecutive a's, then the a on the left must be one of the n numbers to the left of the rightmost n. But that implies that all of the other n's are also in the n numbers to the left of the rightmost n, a contradiction. This shows that the limit inferior of the density of a's left of position p_k is at least $1/2k$.

But now notice that the total density is always greater than 1: we have that the limit inferior of the density approaches

$$\sum_{i=1}^{k-1}\frac{1}{2k}+\sum_{i=k}^{n}\frac{1}{k}=\frac{k-1}{2k}+\frac{n-k+1}{k}\ge\frac{k-1}{2k}+\frac{k-1}{k}=\frac{3k-3}{2k},$$

which is greater than 1 when $k\ge4$, a contradiction. In the cases where $k=2$ or 3, note that we have the following cases:

- Case $k=2$, $n=3$: In this case note that the limit inferior of the total density is actually $1/4+1/2+1/2=5/4$, a contradiction.
- Case $k=3$, $n=4$: Here, the limit inferior of the density is actually 1, suggesting that it may be possible. However, this would mean that there is a 4 between any two consecutive 3's, and sometimes also a 1. But this is a contradiction, since $1 \equiv 4 \pmod 3$.
- Case $k=3$, $n=5$: In this case note that the limit inferior of the total density is actually $1/6+1/6+1/3+1/3+1/3=4/3$, a contradiction. $\square$

Problem 5.

(proposed by Tony Wang)

A positive integer is a *non-trivial perfect power* if it can be expressed as n^k where n and k are positive integers and $k > 1$. Show that there exist arbitrarily large consecutive square numbers with no other non-trivial perfect powers between them.

Notes on Marking. For this problem, we awarded one mark for the idea of a density argument, and three points for counting the perfect powers. There were many solutions that did these but did not quite finish; in most of these cases we awarded seven points. We were relatively lenient with the actual computation. We did not award points for attempts at constructive arguments, since we don't know of any solution they would lead to.

Solution 1.

(solution by Joe Devine and Tony Wang)

To show this, we will prove that the number of square numbers grows faster than the number of all other non-trivial perfect powers combined. We count how many such numbers there are which are at most n.

Observe that there are $\lfloor\sqrt{n}\rfloor$ square numbers that are at most n. We also count the perfect powers: there are $\sum_{i=3}^{\infty}\left(\lfloor n^{1/i}\rfloor - 1\right)$ other perfect powers less than n^2. So it suffices to prove that the quantity

$$\lfloor\sqrt{n}\rfloor - \sum_{i=3}^{\infty}\left(\lfloor n^{1/i}\rfloor - 1\right)$$

is unbounded.

To this end, observe that $\lfloor n^{1/i}\rfloor = 1$ for $i > \log_2 n$, which means we can instead just look at the finite sum

$$\lfloor\sqrt{n}\rfloor - \sum_{i=3}^{\log_2 n}\left(\lfloor n^{1/i}\rfloor - 1\right)$$

which we can then bound below:

$$\lfloor\sqrt{n}\rfloor - \sum_{i=3}^{\log_2 n}\left(\lfloor n^{1/i}\rfloor - 1\right) \geq \lfloor\sqrt{n}\rfloor - \left(\lfloor n^{1/3}\rfloor - 1\right)\log_2 n.$$

We are then done, since by comparing logarithmic and polynomial growth, there is some constant $c \in (0,1)$ such that $n^{1/3}\log_2 n < c\sqrt{n}$ for all sufficiently large n. $\square$

Problem 6.

(proposed by Joe Devine and Tony Wang)

A set of points in the plane is called *rigid* if each point is equidistant from the three (or more) points nearest to it.

(a) Does there exist a rigid set of 9 points?

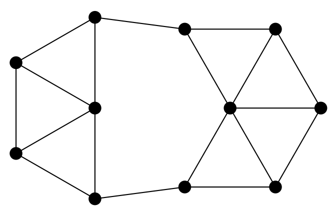
(b) Does there exist a rigid set of 11 points?

Notes on Marking. Unfortunately, no marks were awarded for answering either question correctly. We awarded two marks for solutions to part (b) and eight marks to solutions to part (a). We also awarded one mark for constructions for part (b) which were somewhat close to a working solution, as well as a mark for correct ideas for an initial approach for part (a). Although the proof provided below for part (a) is very long, We did not require the level of detail presented in the proof, and we awarded marks generously to outlines of proofs that could be turned into a full solution.

Solution 1.

(solution by Tony Wang)

We will answer part (a) in the negative, and part (b) in the affirmative. The construction for part (b) follows, so we will focus on part (a).



For part (a), we begin by introducing some terms. We will think of the set of rigid points as a graph, with a directed edge (represented as a directed line segment) *pointing* from each vertex to its three or more closest neighbours. Let the distance from each point P to its closest neighbours be called its *isolation*, and denote it as $I(P)$.

Lemma (Restriction lemma)**.** If all points with isolation greater than some real constant C are removed from a rigid set – that is, if the set of points is *restricted* to points with isolation at most C – the remaining points will still form a rigid set.

Proof. Note that if $I(P) > I(Q)$, then P cannot point to Q, as otherwise Q would also be a distance of $I(P)$ away from P. Hence, after a restriction, all remaining vertices must point to other vertices that still remain, so the out-degree of each remaining vertex does not change and hence the set of points is still rigid. □

We will now also define and prove some lemmas about the convex hull of the set of points. Define a point to be a *hull point* if it is on a vertex or edge of the convex hull. The set of hull points forms the *hull polygon*, and note that, by our definition, it may have internal angles of 180°. The *hull degree* of a hull point is the number of other hull points it points to.

Lemma (Equal isolation lemma)**.** If all points in a rigid set have equal isolation, then it cannot have fewer than 6 hull points.

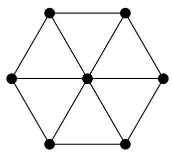
Proof. WLOG, we may assume all points have unit isolation. By pigeonhole principle, the smallest internal angle in the hull polygon must be less than $120°$. However, it is impossible to fit 3 points on an arc with unit radius subtending less than $120°$ without at least one of those points having less than unit isolation. □

Lemma (Hull polygon lemma)**.** A hull polygon cannot be rigid in itself.

Proof. By the restriction lemma, it suffices to prove this lemma for rigid sets of points with equal isolation, since, by restricting a rigid hull polygon to its points of smallest isolation, we obtain another rigid hull polygon, but now with equal isolation.

Now, note that by an argument similar to above, each internal angle must be at least $120°$. Hence the hull polygon must be large enough to contain a regular hexagon of side length 1, and moreover, this regular hexagon can be placed so that it shares a vertex and internal angle bisector with some point P of the hull polygon. But then, letting Q and R be P's neighbours, there is no longer enough space on the minor arc QR with centre P to fit a third point so that it is outside the hexagon and distance at least 1 from Q and R. □

We will now present a proof that, for $n \leq 9$ points, the only rigid set up to dilation and rotation is the *wheel* configuration:



We will use strong induction on the number of points n, and for each case in the table below we will use the arguments indicated by the letters.

		Total points (n)					
		4	5	6	7	8	9
Hull points (k)	3	X+A	X+A	X+A	X+A	Y+A	Z+A
	4	B	X+A	X+A	X+A	Y+A	Z+A
	5		B	X+A	X+A	Y+A	Z+A
	6			B	X+C	Y+D	Z+D
	7				B	Y+E	Z+F
	8					B	Z+E
	9						B

Argument X If the n points were not equally isolated, then we could restrict the set to points of the smallest isolation, yielding a rigid set with less than n points, a contradiction. Hence, all n points must be equally isolated.

Argument A Since all n points are equally isolated, and $k < 6$, we can apply the equal isolation lemma.

Argument B We are done by the hull polygon lemma.

Argument C Since all n points are equally isolated, we can use a similar argument as in the proof of the restriction lemma to show that all internal angles must be exactly $120°$, and hence the hull polygon must be a regular hexagon. The seventh point must therefore be in the centre of the hexagon and we construct the wheel configuration.

Argument Y If a rigid set of 8 points is not equally isolated, then its restriction to the points of smallest isolation must be a wheel, and hence the eighth point must be outside the wheel. However, the circumcentre of any three hull points is the centre of the wheel.

Argument D If we use a non-regular equiangular hexagon as the hull polygon, consider two adjacent sides of differing length. Since the vertex they both connect to has interior angle $120°$, one of the non-hull-points will need to be on the angle bisector, and another will need to be on the longer side, but this implies there are actually more than 6 hull points. If we use a regular hexagon, Argument C shows that there is only one more point that can fit inside the hexagon.

Argument E All k hull points must point to its two neighbours as well as the non-hull-point, but that implies all hull points lie on a circle centred at the non-hull-point, a contradiction since the side length of a regular k-gon is smaller than its radius.

Argument Z If a rigid set of 9 points is not equally isolated, then its restriction to the points of smallest isolation must also be a wheel. Call the non-wheel points P and Q. Since a wheel is convex, P and Q cannot point to 3 points on the wheel, so they must point to 2 points on the wheel and to each other. This implies they must lie on the perpendicular bisector of one of the edges of the wheel. However, since P and Q will be equidistant from edge of the wheel, P, Q, and the center of the wheel will form an equilateral triangle, so P and Q will always be closer to the hull points of the wheel than to each other. Hence, a rigid set of 9 points must be equally isolated.

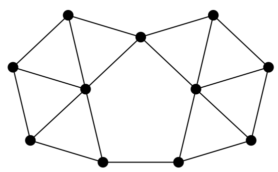
Argument F Note that the 7 hull points must lie on the union of two arcs of unit radius (whose centres are precisely the two non-hull-points). For each hull point of hull degree 1, at least one of its two neighbouring hull points must also have hull degree 1. However, such points need to point to both non-hull-points, and so they can only be placed where the two arcs intersect, which presents a contradiction. Hence, all hull points have hull degree 2, implying that the hull polygon must be equilateral.

Now, since the distance between the non-hull points must be at least 1, the perimeter of the convex hull of the two arcs must have total distance at least $2 + 2\pi$. The greatest length of this perimeter that can be covered by a unit disc whose centre is a hull point is $2\pi/3$, and if all hull points point to both their neighbours, each point on the perimeter must be covered twice. However, this means that the total length of perimeter coverage is at most $7 \times 2\pi/3 \approx 14.66$, but this is still smaller than twice the perimeter, which is $4 + 4\pi \approx 16.56$. □

Comment. It is relatively easy to construct rigid sets with 10 or more than 11 points, meaning that we can conclude that rigid sets exist precisely when n is 7 or greater than 9.

Solution 2. *(solution by Christina Dong)*

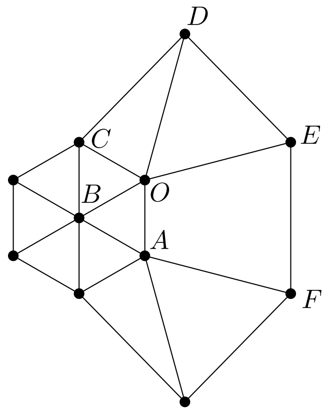
We present an alternative construction for part (b), where each line segment has length 1:



□

Solution 3. *(solution by Tian Hao [Cambridge])*

We present an alternative construction for part (b), where each line segment has either length 1 or 2. To prove that a length of 2 works, consider that if $DC = DO = DE = EO$, then if we rotate ABD $60°$ clockwise around O we get BCE, and hence CE is always horizontal. Using a similar argument for F, this implies that EF is always equal to the height of the hexagon, that is, 2.



$\square$