IMPERIAL-CAMBRIDGE
MATHEMATICS
COMPETITION

ICMC 8 — Round Two

23 February 2025

Name: _____

Contestant ID: _____

University: _____

Instructions:

- Do not turn over until told to do so.
- You will have 4 hours to solve 5 problems, each of which carries 10 marks.
- Use a black or blue pen or a dark pencil. Rulers, compasses, protractors, and erasers may be used but will not be required. All electronic devices, including calculators, are prohibited.
- Drinks are allowed, but food is prohibited.
- Write your solution to each problem on a different page. At the top of each page, write down the question number, your initials, and your contestant number. Use both sides whenever possible. Write clearly and not too faintly – your work will be scanned for marking.
- Problems are listed roughly in order of difficulty. Proofs are expected for all problems even if they only ask for an answer.
- One complete solution will be awarded more marks than several unfinished attempts.
- You may not leave the contest venue in the first two hours or the last thirty minutes unless exceptional circumstances arise.
- You may take away the problems sheet and any rough work when leaving the venue.

Problem 1. A cube of side length 2025 is dissected into cubes of side length 2 and cubes of side length 1. What is the minimum number of cubes of side length 1?

Problem 2. Given a line k and an acute triangle ABC, show how to construct using straightedge and compass a line ℓ parallel to k such that ℓ splits the perimeter of ABC in half.

Problem 3. Do there exist positive integers $a, b, c < 225$ such that, for the quadratic $f(x) = ax^2 + bx + c$, the sequence 0, $f(0)$, $f(f(0))$, $f(f(f(0)))$, ..., leaves every possible remainder when divided by 225?

Problem 4. A function $f : [0, 1] \to \mathbb{R}$ is *chromatic* if:

- for all $x, y \in [0, 1]$, $|f(x) - f(y)| \leq |x - y|$, and
- $\int_0^1 f(x)\,\mathrm{d}x = 1/2$.

Over all pairs $f, g : [0, 1] \to \mathbb{R}$ of chromatic functions, what is the minimum value of

$$\int_0^1 f(x)g(x)\,\mathrm{d}x?$$

Problem 5. Let an $n \times n$ matrix be called *bionic* if each entry is either 0 or 1, no two rows are the same, and no two columns are the same. Given a bionic matrix, a *move* consists of either

- reading the rows of the matrix as binary numbers, and reordering them from largest to smallest so that higher rows have larger numbers; or
- reading the columns of the matrix as binary numbers, and reordering them from largest to smallest so that columns further to the left have larger numbers.

A move is only *valid* if it results in a change to the matrix. For example, the following represents a valid sequence of two moves on a 3×3 bionic matrix:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Over all $n \times n$ bionic matrices, find the length of the longest valid sequence of moves in terms of n.