IMPERIAL-CAMBRIDGE MATHEMATICS COMPETITION

8th Edition (2024–2025)

ROUND TWO

Official Solutions*

Last updated: 28 Mar 2025, 9pm

*A solution may receive full or partial marks even if it does not appear in this booklet.

Problem 1. *(proposed by Tony Wang)*

A cube of side length 2025 is dissected into cubes of side length 2 and cubes of side length 1. What is the minimum number of cubes of side length 1?

Notes on Marking. Most full credit solutions used arguments similar to Solution 1 below. For solutions of this type, 1 mark was awarded for using the equivalence of maximal number of 2-cubes and minimal number of unit-cubes, 3 marks were awarded for carefully defining a colouring, 4 marks were awarded for using the defined colouring to find the maximal number of 2-cubes to be 1012^3, 1 mark was awarded for stating that 1012^3 2-cubes tiled a 2024-cube, and finally, 1 mark was awarded for writing down a correct numerical expression for the minimal number of **unit** cubes (which did not need to be simplified).

No marks were awarded for proving that the dissection must be parallel to and/or at integer distances to the faces of the cube.

A handful of contestants tried using compacting arguments similar to those described in Solutions 2 and 3 to argue that no more than 1012^3 2-cubes can be tiled into the 2025-cube. These attempts were marked individually based on the thoroughness of the arguments provided.

Solution 1. *(solution by Tony Wang)*

We prove that the minimum number of unit cubes is $2025^3 - 2024^3$. Note that this is possible since we can tile a 2024-cube with 1012^3 2-cubes, and we can fit precisely $2025^3 - 2024^3$ unit cubes in the remaining space. Hence, it suffices to prove that $2025^3 - 2024^3$ is indeed the minimum.

Denote cubes of side length 2 as 2-cubes. First, note that the minimum number of unit cubes can be found by equivalently finding the maximum number of 2-cubes. To find the maximum number of 2-cubes, Then, aligning the cube with the x–y–z axes and dividing it into 2025^3 unit cubic cells, such that one of the corner cells is assigned the coordinates $(1, 1, 1)$, ensuring that each cell has integer coordinates.

Colouring every cell with coordinates in $\{(2k, 2l, 2m) : k, l, m \in \mathbb{Z}\}$, we note that there are exactly 1012^3 coloured cells in the big cube and that every lattice-aligned cube of side length 2 contains exactly one coloured cell. This means that there can be at most 1012^3 cubes of side length 2. This is clearly a possible dissection of the sub-cube of side length 2024). Hence, the minimum number of unit cubes is $2025^3 - 2024^3$, as desired. □

Solution 2. *(solution by Gergely Rozgonyi)*

As above, the goal is to prove that any number of 2-cubes fitting into the 2025-cube (with unit cubes filling the rest of the space) can be rearranged to fit within a 2024-cube. For the purposes of rearranging the 2-cubes, we can replace all the unit cubes with empty space.

Aligning the cube with the x–y–z axes, we denote by the *x-distance* of a 2-cube the distance from the $x = 0$ face of the 2025-cube to the 2-cube's nearest parallel face, and define the y- and *z-distances* similarly. Note that in the full 2024-packing, all 2-cubes have even values for each of these three special distances. Furthermore, any arrangement of 2-cubes where each of these has even special distances necessarily implies that no 2-cube falls (partially) outside the 2024-cube: the only situation where they lie outside is when one of the special distances (bounded above by 2023) is greater than 2022. First, "compact" the cube in (WLOG) the x-direction:

- Pick any 2-cube that has a smallest **odd** x-distance – if there are multiple pick any one of them;

- For any such 2-cube, the face closer and parallel to $x = 0$ necessarily has a $1 \times 2 \times 2$ gap in the 2025-cube with the unit cubes removed: if it had not, any 2-cube intersecting this space must touch the chosen 2-cube, which would imply that it has a smaller odd x-distance, and our original choice did not satisfy the criterion of having smallest odd x-distance;
- Translate the chosen 2-cube one unit closer to the x-axis.

This process satisfies the following:

- In each step, the number of 2-cubes with odd x-distance decreases by 1. Note that the number of these cubes is bounded above by 1013^3, so the process must eventually come to an end;
- A cube with minimal odd x-distance exists unless all cubes have even x-distance: the minimal odd distance is bounded below by 1.

Therefore, the process has a certain end point, at which all 2 cubes have even x-distance, so they fit into a $2024 \times 2025 \times 2025$ cube. Repeating the analogous processes for y- and z-distances we "compacted" all of the 2-cubes into a 2024-cube. As this can be dissected trivially into a dense tiling of 2-cubes, we have that the maximal number of 2-cubes is 1012^3 and the minimal number of unit cubes required for the dissection is $2025^3 - 2024^3$. $\square$

Solution 3. *(solution by contestants)*

We will once again show that no more than 1012^3 2-cubes can be compacted into the 2025-cube, with the rest of the arguments required for full credit discussed in the above solutions.

Align the cube the same way as above, and replace all unit cubes with empty space. Next, define an *x-compression* as applying gravity in the negative x direction (so that cubes get "pushed" towards the $x = 0$ plane), and define y- and *z-compressions* similarly. Now, repeat the following process until we are left with a $1 \times 2025 \times 2025$ cuboid:

1. Perform an $\underline{x\text{-compression}}$;
2. Slice off the $\underline{2 \times 2025 \times 2025 \text{ cuboid}}$ closest to the $\underline{x = 0}$ plane. Note that this will never cut any 2-cubes in half, as that would mean we have a 2-cube "hovering" over empty space with gravity applied.

After 1012 iterations of this process, we are left with a cuboid of unit thickness, which naturally cannot fit any 2-cubes. Note also that the maximal number of 2-cubes must then be $1012\times$the maximal number of 2-cubes in a sliced off $2 \times 2025 \times 2025$ cuboid.

To determine the maximal number of 2-cubes in the slices, we perform an analogous algorithm starting from the $2 \times 2025 \times 2025$ slice, but with $\underline{y\text{-compressions}}$ and $\underline{2 \times 2 \times 2025 \text{ cuboid}}$ slices until we are left with a $2 \times 1 \times 2025$ cuboid. This process will also end after 1012 iterations and the final cuboid of unit thickness must once again fit no 2-cubes. Then, the maximal number of 2-cubes in the original cube must be $1012^2\times$the maximal number of 2-cubes in a $2 \times 2 \times 2025$ cuboid. This latter – through a similar argument with z-compressions, or otherwise – is 1012, and the maximal number of 2-cubes contained in the 2025-cube is 1012^3. $\square$

Problem 2. *(proposed by Tony Wang)*

Given a line k and an acute triangle ABC, show how to construct using straightedge and compass a line ℓ parallel to k such that ℓ splits the perimeter of ABC in half.

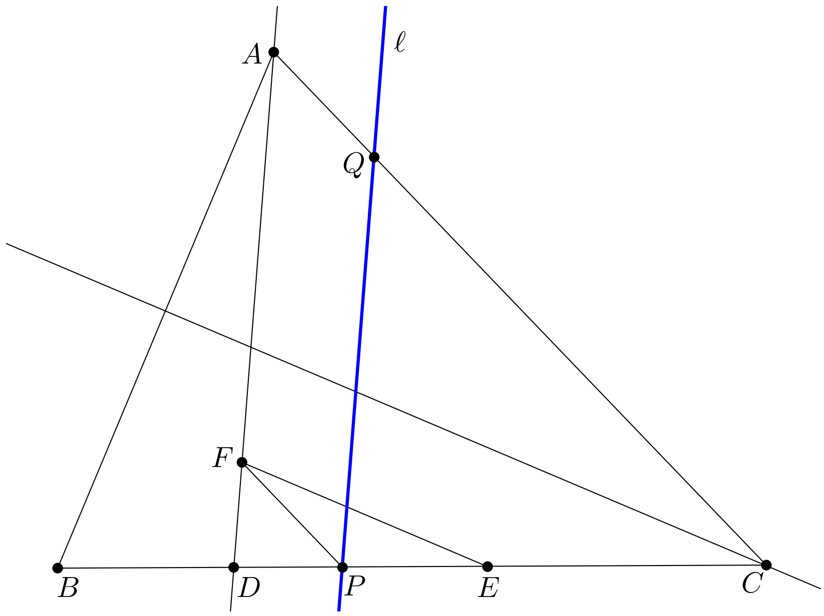
Notes on Marking. Some people lost marks for not dealing with the case where k is parallel to one of the sides of the triangle. Other similar neglect of edge cases were also penalised.

Solution 1. *(solution by Tony Wang)*

We will assume basic straightedge and compass constructions in this solution.

First, note that at least one of the lines parallel to k passing through the three vertices of the triangle must intersect the triangle at more than one point. Suppose without loss of generality that the line parallel to k passing through A does this. Let the line intersect the line segment BC at D. Then, using the compass, measure out the perimeter p of the triangle, and then measure a point E along the perimeter of the triangle starting from A with distance $p/2$. Note that, by our construction, AE bisects the perimeter of ABC.

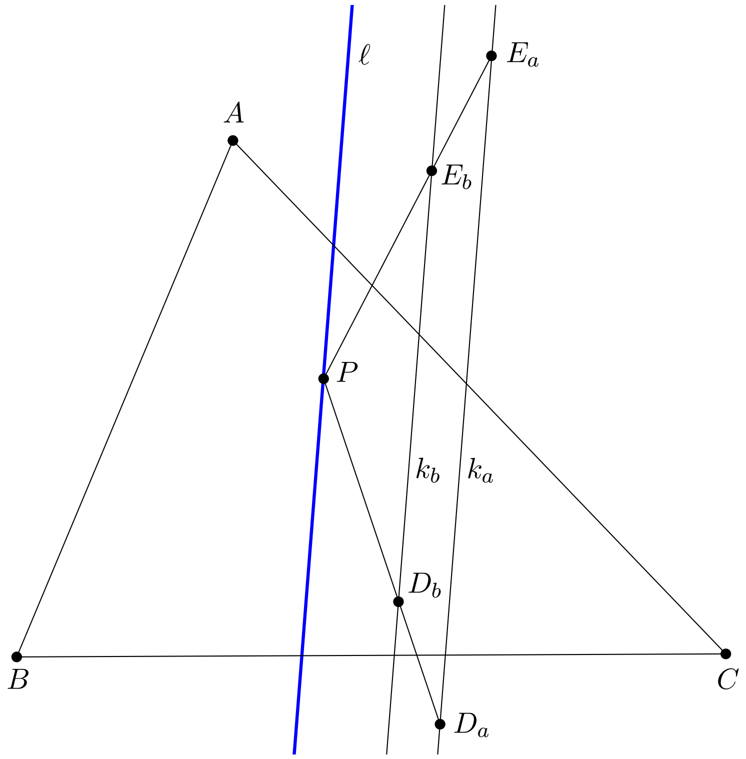
If $D = E$, then we are done. Otherwise, suppose without loss of generality that C lies on the ray DE. Then we know that ℓ must be closer to C than AD. Hence, it suffices to find points P on DC and Q on AC such that $PQ \parallel \ell$ and $AQ = EP$. To construct P, let m be the internal angle bisector of $\angle BCA$. Let the line parallel to m passing through E intersect AD at F. Then the line passing through F parallel to AC intersects BC at P. Of course, we can then construct Q by drawing a line through P parallel to ℓ and intersecting it with AC.



It remains to prove that $AQ = EP$. Letting the angle at C be $2c$, we note that $\angle PEF = c$ as $EF \parallel m$. Now since $FP \parallel AC$, then $\angle FPD = \angle ACB = 2c$, and hence we have $\angle PFE = c$ as well. This implies that triangle PFE is isosceles, and so $FP = EP$. However, since $AQPF$ is a parallelogram, we also have $FP = AQ$, and so we are done. $\square$

Solution 2. *(solution by Casper Madlener (Leiden) and Paul Hametner (Bristol))*

In this solution all lengths of line segments are directed. Suppose that we know that ℓ lies intersects line segments AB and BC, as in the previous solution. Now draw two lines k_a and k_b, both parallel to k and intersecting AC and BC. Let D_a be any point of k_a, and construct E_a on k_a so that the line segment D_aE_a is the length of the perimeter of ABC to the left of k_a minus the perimeter of ABC to the right of k_a. Define D_b and E_b similarly. (In particular, we make sure to account for the possibly negative lengths of line segments.) Now, let P be the intersection of D_aD_b and E_aE_b. Then we claim that the line through P parallel to k is ℓ.



This is true because, as we vary a line k_x parallel to k over the triangle (where x is the displacement of k_x from k), the difference in perimeter on the left and right sides of k_x varies linearly in x, as long as k_x still intersects the line segments AC and BC. Hence, the lines D_aD_b and E_aE_b have the property that any other k_x which intersects both AC and BC will yield $D_x \coloneqq D_aD_b \cap k_x$ and $E_x \coloneqq E_aE_b \cap k_x$ such that D_xE_x represents the difference in perimeter on the left and right sides of k_x. Hence, the line parallel to k which passes through P must have such a difference of 0. That is, the line must bisect the perimeter of ABC.

Problem 3.

(proposed by Dylan Toh)

Do there exist positive integers $a, b, c < 225$ such that, for the quadratic $f(x) = ax^2 + bx + c$, the sequence 0, $f(0)$, $f(f(0))$, $f(f(f(0)))$, $\ldots$, leaves every possible remainder when divided by 225?

Notes on Marking. We awarded two marks for coming up with a correct choice of a, b, c without proof, and one mark for narrowing down the possible values enough that you conceivably could have guessed one. Ideas towards the correct solution, such as setting $b = 1$, were also awarded two marks. We deducted one mark for lack of justification that if all possible values modulo 9 and all possible values of 25 are achieved then we have done so modulo 225; there were actually two instances of the Chinese remainder theorem in this argument (the justification people often forgot to write was using the fact that the periods of the two sequences are coprime).

Solution 1.

(solution by contestants)

We claim that $a = 150$, $b = 1$ and $c = 1$ works. Let (x_n) be the sequence defined by $x_0 = 0$ and $x_n = f(x_{n-1})$ for $n > 1$.

Firstly, we observe that $f(x) \equiv x+1 \pmod{25}$ for all x, which means $x_n \equiv n \pmod{25}$ for all n. Also we observe that the first ten terms of the sequence modulo 9 are $0, 1, 8, 6, 7, 5, 3, 4, 2, 0$. Since each term of the sequence depends only on the previous term, we see that it is a repeating sequence of all nine residues modulo 9 (and has a period of nine).

Using the Chinese remainder theorem, since 9 and 25 are coprime, we know that if two values are the same modulo $9 \times 25 = 225$ then they must be the same modulo 9 and modulo 25. This means that we are done if every possible pair of values modulo 9 and modulo 25 is achieved by the sequence. However, this is indeed true, since the periods of the sequence modulo 9 and modulo 25 are coprime, so using the Chinese remainder theorem again we get the desired result. □

Problem 4.

(proposed by Ishan Nath)

A function $f : [0, 1] \to \mathbb{R}$ is *chromatic* if:

- for all $x, y \in [0, 1]$, $|f(x) - f(y)| \leq |x - y|$, and
- $\int_0^1 f(x)\,\mathrm{d}x = 1/2$.

Over all pairs $f, g : [0, 1] \to \mathbb{R}$ of chromatic functions, what is the minimum value of

$$\int_0^1 f(x)g(x)\,\mathrm{d}x?$$

Notes on Marking. 1 mark was given for getting the correct answer and exhibiting a pair (f, g) that achieve $1/6$, and 2 marks were given for showing that we could restrict to $g = 1 - f$ when bounding the minimum. No marks were given for proving inequalities about the range of chromatic functions, or considering $f - 1/2$, $g - 1/2$. Solutions that restricted to $f = 1 - g$ and then considered the non-decreasing rearrangement of f often failed to show that this rearrangement still satisfied the first condition for being chromatic.

Solution 1.

(solution by Ishan Nath and Samuel Liew)

We will show that the answer is $1/6$.

We can obtain this bound with $f(x) = x$ and $g(x) = 1 - x$. We can easily check that these satisfy the two conditions required to be chromatic, and

$$\int_0^1 f(x)g(x)\,\mathrm{d}x = \int_0^1 x - x^2\,\mathrm{d}x = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$

Now we show that this bound is optimal. For f and g chromatic, consider the integral

$$I = \int_0^1 \int_0^1 (f(x) - f(y))(g(x) - g(y))\,\mathrm{d}x\,\mathrm{d}y.$$

Since we know the integrals of f and g, we can expand and evaluate this as

$$\begin{aligned} I &= 2\int_0^1 \int_0^1 f(x)g(x)\,\mathrm{d}x\,\mathrm{d}y - 2\int_0^1 f(x)\,\mathrm{d}x \int_0^1 g(y)\,\mathrm{d}y \\ &= 2\int_0^1 f(x)g(x)\,\mathrm{d}x - \frac{1}{2}. \end{aligned}$$

However, by the first condition, this is bounded:

$$\begin{aligned} |I| &\leq \int_0^1 \int_0^1 |f(x) - f(y)|\,|g(x) - g(y)|\,\mathrm{d}x\,\mathrm{d}y \leq \int_0^1 \int_0^1 |x - y|^2\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1 \int_0^1 (x^2 - 2xy + y^2)\,\mathrm{d}x\,\mathrm{d}y = 2\int_0^1 x^2\,\mathrm{d}x - 2\left(\int_0^1 x\,\mathrm{d}x\right)^2 \\ &= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}. \end{aligned}$$

So $I \geq -1/6$, hence combining these two expressions for I we get

$$\int_0^1 f(x)g(x)\,\mathrm{d}x \geq \frac{1}{2}\left(\frac{1}{2} - \frac{1}{6}\right) = \frac{1}{6}. \qquad \square$$

Comment. We can show that equality holds only if f and g are both in $\{x, 1-x\}$. This shows that the unique minimizer is $\{f, g\} = \{x, 1-x\}$.

Solution 2. *(solution by contestants)*

We provide an alternative proof that the lower bound is correct. We first argue that we may consider only $g = 1 - f$ when bounding the minimum.

First notice that if g is chromatic, then so is $1-g$. Indeed, we can verify that both conditions hold. Then,

$$\int f(1-g)\,\mathrm{d}x = \int f\,\mathrm{d}x - \int fg\,\mathrm{d}x = \frac{1}{2} - \int fg\,\mathrm{d}x.$$

So bounding the inner product of (f, g) from below is equivalent to bounding the inner product of $(f, 1-g)$ from above. From now on, we consider the problem of bounding the inner product from above.

We argue that to bound the inner product from above, it suffices to consider $f = g$. Indeed, from the Cauchy-Schwarz inequality,

$$\int fg\,\mathrm{d}x \le \left(\int f^2\,\mathrm{d}x\right)^{1/2} \left(\int g^2\,\mathrm{d}x\right)^{1/2}.$$

So if we bound the inner product of (f, f) and (g, g) above by A, then the inner product of (f, g) must be at most A as well. Now, we note that

$$\begin{aligned}\int_0^1 \big(f(x) - f(1/2)\big)^2\,\mathrm{d}x &= \int_0^1 f(x)^2\,\mathrm{d}x - 2f(1/2)\int_0^1 f(x)\,\mathrm{d}x + f(1/2)^2 \\ &= \int_0^1 f^2\,\mathrm{d}x - f(1/2) + f(1/2)^2,\end{aligned}$$

but also

$$\int_0^1 \big(f(x) - f(1/2)\big)^2\,\mathrm{d}x \le \int_0^1 (x-1/2)^2\,\mathrm{d}x = \left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}.$$

So,

$$\int_0^1 f^2\,\mathrm{d}x \le \frac{1}{12} + f(1/2) - f(1/2)^2 \le \frac{1}{12} + \frac{1}{2} - \frac{1}{4} = \frac{1}{3}.$$

This follows as the function $y - y^2$ is maximized at $y = 1/2$. This shows that the inner product of $(f, 1-g)$ is bounded above by $1/3$, and hence

$$\int fg\,\mathrm{d}x = \frac{1}{2} - \int_0^1 f(1-g)\,\mathrm{d}x \ge \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$ □

Comment. Alternatively, instead of using Cauchy-Schwarz we can use AM-GM:

$$\int_0^1 fg\,\mathrm{d}x \le \int_0^1 \frac{f^2+g^2}{2}\,\mathrm{d}x = \frac{1}{2}\int_0^1 f^2\,\mathrm{d}x + \frac{1}{2}\int_0^1 g^2\,\mathrm{d}x \le \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

Solution 3. *(solution by Dylan Toh)*

We prove the lower bound using integration by parts. First, chromatic functions are 1-Lipschitz, and Lipschitz functions are known to satisfy the fundamental theorem of calculus: if f is 1-Lipschitz, then it is almost-every differentiable with $|f'| \leq 1$, and for all $0 \leq a < b \leq 1$,

$$f(b) - f(a) = \int_a^b f'(x)\,\mathrm{d}x.$$

For convenience, we shift f and g down by $1/2$, i.e. redefine $f_1 = f - 1/2$, $g_1 = g - 1/2$. Then f_1, g_1 both still satisfy the first condition for being chromatic, but now have integral 0. Then notice that

$$\int_0^1 fg\,\mathrm{d}x = \int_0^1 f_1 g_1\,\mathrm{d}x + \frac{1}{4},$$

so it suffices to lower-bound the inner product of (f_1, g_1), over all 1-Lipschitz functions which have integral 0.

From now on, we work with f_1, g_1, and omit the subscript. Define

$$G(x) = \int_0^x g(y)\,\mathrm{d}y.$$

Then using integration by parts,

$$\int_0^1 fg\,\mathrm{d}x = \big[fG\big]_0^1 - \int_0^1 f'G\,\mathrm{d}x = -\int_0^1 f'G\,\mathrm{d}x \geq -\int_0^1 |G|\,\mathrm{d}x,$$

where we use the fact that $G(0) = G(1) = 0$, as g integrates to 0, and $|f'| \leq 1$ almost-everywhere since f is 1-Lipschitz. Hence it suffices to upper-bound the integral of $|G|$. We claim that

$$|G(x)| \leq \frac{x(1-x)}{2}.$$

This follows from:

$$\begin{aligned}
G(x) &= \int_0^x g(y)\,\mathrm{d}y = \int_0^x g(x)\,\mathrm{d}y + \int_0^x \big(g(y) - g(x)\big)\,\mathrm{d}y \\
&= xg(x) + \int_0^x \big(g(y) - g(x)\big)\mathrm{d}y, \\
G(x) &= -\int_x^1 g(y)\,\mathrm{d}y = -\int_x^1 g(x)\,\mathrm{d}y - \int_x^1 \big(g(y) - g(x)\big)\,\mathrm{d}y \\
&= (x-1)g(x) - \int_x^1 \big(g(y) - g(x)\big)\mathrm{d}y.
\end{aligned}$$

Subtracting x times the second equation from $x-1$ times the first,

$$\begin{aligned}
G(x) &= -x\int_x^1 (g(y) - g(x))\,\mathrm{d}y - (x-1)\int_0^x (g(y) - g(x))\,\mathrm{d}y \\
\implies |G(x)| &\leq x\int_x^1 |y - x|\,\mathrm{d}y + (1-x)\int_0^x |y - x|\,\mathrm{d}y \\
&= \frac{1}{2}\big[x(1-x)^2 + (1-x)x^2\big] = \frac{x(1-x)}{2}.
\end{aligned}$$

From this, we can bound:

$$-\int_0^1 |G|\,\mathrm{d}x \geq -\int_0^1 \frac{x(1-x)}{2}\,\mathrm{d}x = -\left[\frac{x^2}{4} - \frac{x^3}{6}\right]_0^1 = -\frac{1}{12}.$$

This gives a lower bound for the inner product of f, g when they have integral 0. Returning to the integral $1/2$ case, we find

$$\int_0^1 fg\,\mathrm{d}x = \int_0^1 f_1 g_1\,\mathrm{d}x + \frac{1}{4} \geq -\frac{1}{12} + \frac{1}{4} = \frac{1}{6}.$$

□

Problem 5. *(proposed by Tony Wang)*

Let an $n \times n$ matrix be called *bionic* if each entry is either 0 or 1, no two rows are the same, and no two columns are the same. Given a bionic matrix, a *move* consists of either

- reading the rows of the matrix as binary numbers, and reordering them from largest to smallest so that higher rows have larger numbers; or
- reading the columns of the matrix as binary numbers, and reordering them from largest to smallest so that columns further to the left have larger numbers.

A move is only *valid* if it results in a change to the matrix. For example, the following represents a valid sequence of two moves on a 3×3 bionic matrix:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Over all $n \times n$ bionic matrices, find the length of the longest valid sequence of moves in terms of n.

Notes on Marking. We did not award any marks to contestants who proved other upper bounds on the number of moves, if it did not contribute to the proof of the true upper bound.

Solution 1. *(solution by Tony Wang)*

We will prove that the answer is $2n - 3$ for $n > 2$, 2 for $n = 2$, and 0 for $n = 1$.

Notate the cell in the i-th row and j-th column as (i, j). If a row R has 1's in every position that another row S has 1's, plus more, then we say that row R *majorises* row S, and note that this implies that row R is always larger than row S since majorisation is independent of the order of the columns or rows. A similar terminology applies to columns.

Lemma 1: on each row move, at least one more row becomes *locked* and it's position will never change in the future (even if the entries in the row change order); and similarly for columns.

Proof. We will prove this by induction.

- <u>Base case:</u> Suppose that we have performed no row moves so no rows are locked. After the first row move, we will prove that the first row, R, becomes locked. If at any future point, some other row S becomes greater than R so that S would become the first row, then note that this requires that the first difference between R and S (when read from the left) is a 0 in R and a 1 in S. Define D to be the column where R and S are different, and d to be the position of row D, so that these two cells are at positions $(1, d)$ and (s, d). However, the move before must have been a column move, and hence $(1, d) = 0$ implies that $(1, i) = 0$ for all $i > d$. But this implies that S majorises R. Hence, on the very first move, S must have been greater than R, a contradiction. A similar argument applies to columns.
- <u>Inductive step:</u> Suppose that the first $r - 1$ rows are locked and the r-th row is not locked for some $r \geq 1$, and we perform another row move. We want to prove that after this move, the r-th row, which we call R, also becomes locked. Firstly, since row moves and

column moves must be interlaced, we must have just performed a column move. Define S, s, D, and d similarly to the base case, and suppose that at some future point, row S takes the place of row R. Then we must have $(r, d) = 0$ and $(s, d) = 1$. Define C to the leftmost column where R differed from S when R became locked, and let the position of column C be c. Then we must have $(r, c) = 1$ and $(s, c) = 0$. Clearly, we must have had $c < d$ then and $d < c$ now. If D moved to the left of C due to some pair of entries in the first $r - 1$ rows, then since the first $r - 1$ rows were always locked, this implies that D would have already been to the left of C when R became locked, a contradiction. Hence all the entries in the first $r - 1$ rows of D and C must be equal. But then since $(r, d) = 0$ and $(r, c) = 1$, row R must be the first row where C and D differ, and hence no column move will ever move D to the left of C. A similar argument applies to columns.

Hence, we have proven this lemma by induction. ■

Lemma 2: The second move in any sequence of moves will always lock at least two rows/columns.

Proof. WLOG the first move is a column move, and suppose that after the column move, the first row, R, has c 1's in a row, followed by $m - c$ 0's in a row, where $c \geq 2$. We already know that the next row move will lock row 1, so it remains to prove that it will also lock another row. Suppose that during the first row move, R doesn't change position. In this case, we know that no other row majorises R, and hence we can assume that R was already locked *before* the first column move. Hence the first row move will lock the second row. If instead, R moves to position r, then all of the first $r - 1$ rows must majorise row r, in particular, this means that R will never be in a position less than r. Meanwhile, since there is now a block of $r \times c$ 1's in the top left corner of the matrix, no 0's can be introduced into this block, meaning that row R will always have c 1's in a row followed by $m - c$ 0's. So if it ever drops to a position below row r, that implies the row that replaced R majorised R as well, a contradiction since this would have occurred in the first row move instead of later. Thus, R will also be locked in position r. ■

Using the two lemmas above, and the fact that locking all but one row/column is effectively locking all rows/columns, we can deduce an upper bound of $2n - 3$ on the answer for $n > 2$. For the construction use the following bionic matrix (to be used with a row move first):

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

When $n = 2$, note that if the first move is a column move, then it locks both columns, and the second move will lock both rows. A similar argument applies when the first move is a row move. Hence, the upper bound in this case is 2. We can achieve this using the following bionic matrix:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Finally, in the case where $n = 1$, it is clear that we cannot make any moves. □