IMPERIAL-CAMBRIDGE
MATHEMATICS
COMPETITION

9th Edition (2025–2026)

# ROUND ONE

## Official Solutions*

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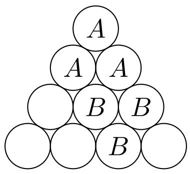
*A solution may receive full or partial marks even if it does not appear in this booklet.

# Problem 1. *(proposed by Andrija Živadinović)*

Tony draws $1 + 2 + 3 + \cdots + 2026$ unit circles arranged in a triangular lattice, forming an equilateral triangular array with 2026 rows. A *triple* consists of 3 mutually tangent circles, and a pair of triples is considered *disjoint* if they do not share a circle. Given a collection of disjoint triples, we call a circle *unused* if it is not in any of the triples. Over all collections of disjoint triples, what is the smallest possible number of unused circles?

(As an example, the diagram below shows a pair of disjoint triples in an equilateral triangular array with 4 rows, leaving 4 circles unused.)

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**Notes on Marking.** The markscheme for this problem awarded 1 mark for proving the lower bound of 1 unused circle, and 9 marks for providing a construction to that effect. A correct answer alone did not receive any marks. Constructions achieving other numbers of unused circles were not also awarded marks, except when they contained ideas that were relevant to one of the many proper constructions.

A common mistake in proving the lower bound was to say that the minimum number of unused circles is 1 because $2026 \equiv 1 \pmod 3$. This was not awarded any marks. Some contestants who miscalculated, or who only stated that it would be possible to arrive at a lower bound by considering the number of circles modulo 3, did not receive this mark.

Many contestants who arrived at Solution 1 lost marks for one or more of the following: (a) not giving a proper description of the tiling in the general case (though we accepted well-labelled diagrams), (b) not showing why the tiling works when the length of the triangle is $6n + 4$, and (c) not correctly dealing with the base case.

Some constructions did not take into account that certain trapezium structures can only be used when the length of the trapezium satisfy certain conditions. For example, a 2-high trapezium can only be tiled when the lower row has $n \equiv 2 \pmod 3$ circles, and a 3-high trapezium can only be tiled when the lowest row has $n \equiv 0 \pmod 2$ circles.

Some contestants attempted to use a 9-, 11-, or 12-row triangles to partition the 2026-row triangle in a grid-like fashion. However, for this to work, two triangles with lengths differing by one would be needed (see the third comment to Solution 2). Some contestants attempted to generate a fractal-like tiling, but this doesn't work either.

A greedy approach can be made to work (see Solution 4), but all the unused circles should always remain all on the left, or all on the right, after each step. While it is true that other patterns of unused circles on a row can always be filled in without clashes on the row below, the markers do not know of a solution which uses this fact.

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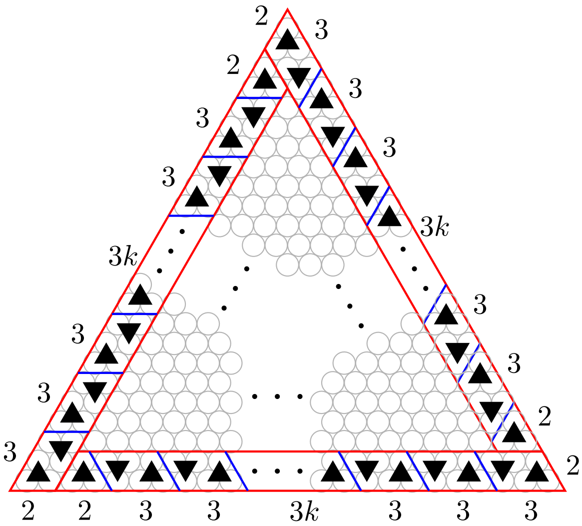
## Solution 1. *(ICMC solution by Tony Wang)*

We prove that the answer is 1. We start by proving that the answer must be at least 1. This is true because the total number of circles in the equilateral triangular array is

$$1 + 2 + \cdots + 2026 = \frac{2026 \times 2027}{2} = 1013 \times 2027,$$

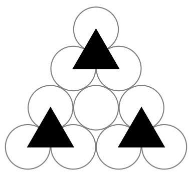
which, modulo 3, works out to be $2 \times 2 \equiv 1 \pmod 3$.

We now need to construct a collection of disjoint triples that use all but one circle. To do this, we note that, given that the side length of the equilateral triangular array is a number congruent to 1 modulo 3, we can use the following construction along the edge:




Note that this construction works whenever the side length of the equilateral triangle is 1 (mod 3) since the triplets along the border have a total length of $3n + 2 + 2 \equiv 1 \pmod 3$.

Using this construction essentially reduces the side length of the equilateral triangle by 6, since it reduces the triangle on each side by 2. By repeating this construction 337 times, we will end up with an equilateral triangular array of side length 4, and we can tile that triangular array as follows:



This leaves 1 circle unused. □

## Solution 2. *(solution by Contestants)*

It is possible to construct a collection of triples which use all the circles in an equilateral triangular array with 12 rows. Also, using two triples, we can use all the circles in a $2 \times 3$ parallelogram. This in turn allows us to use all the circles in a $2a \times 3b$ parallelogram. We can then use up all the circles in the first $2016 = 12 \times 168$ rows of our 2026-row triangle by using a 12-row triangle and an associated 12-high parallelogram to use all the circles in rows $12k + 1, \ldots, 12k + 12$, for $k = 1, \ldots, 168$. For the last 10 rows, we can use a 10-row triangle with one circle missing together with a $10 \times 2016$ parallelogram.

**Comment.** It is also possible to show that we can use up all the circles of any $12 \times k$ parallelogram for all $k \geq 2$. This allows us to start from the bottom up, leaving just the top 10-rows, which we can cover with a 10-row triangle as above.

**Comment.** Another related approach is to use 9-row triangles instead of 12-row triangles, with the rows $12k + 10$, $12k + 11$, and $12K + 12$ above grouped into a single upward-pointing triple and a $3 \times (12k + 10)$ parallelogram.

**Comment.** One further approach, used by Evelyn Ebneter, is to partition each set of 12 rows into upward-pointing 12-row triangles and downward-pointing 11-row triangles. This works precisely because the bottom row of each set has a number of circles divisible by 12.

## Solution 3. *(solution by Contestants)*

As in Solution 1, we will use trapeziums to break up the outer layers of the triangle and reduce the problem. However, we will use 3-high trapeziums instead of 2-high ones. This is only possible when the outer triangle has odd length, and performing this reduces the length by 9. Hence, after this operation, we will be left with a triangle of even length, but we can use a single 3-high trapezium covering the bottom three rows of the triangle to return to a triangle with odd length. Repeating this alternating process 168 times (starting with a the single 3-high trapezium step) brings us to a 10-row triangle, for which we can use all but one circle.

## Solution 4. *(solution by Xu Chen Tan)*

We show that the smallest possible number of unused circles is 1, as in Solution 1. We will define an algorithm for filling in each row in a greedy fashion so that all the unused circles on each row are either all on the left or all on the right of that row. That is, starting from row 1, consider the unused circle to be on the "left side". Then, fill in each row iteratively as follows:

- Suppose we are at row $2k+1$, then the unused circles are on the left side. Starting from the leftmost circles in rows $2k+1$ and $2k+2$, use an upward-pointing triple, and then, staying within rows $2k+1$ and $2k+2$, alternate the triple direction going to the right until the circles on row $2k+1$ are used up. Now, row $2k+2$ will be partially filled on the left, and all the unused circles will be on the right.
- Suppose we are at row $2k$, then the unused circles are on the right side. Starting from the rightmost circles in rows $2k$ and $2k+1$, use an upwards-pointing triple, and then, staying within rows $2k$ and $2k+1$, alternate the triple direction going to the left until the circles on row $2k$ are used up. Now, row $2k+2$ will be partially filled on the right, and all the unused circles will be on the left.

We will now show that this process is well-defined. Firstly, we want to know how many circles will be filled in the next row given that the current row has $n$ unused circles. We claim that this is

$$f(n) = \begin{cases} n & \text{if } n \equiv 0 \pmod 3, \\ n+1 & \text{if } n \equiv 1 \pmod 3 \end{cases}.$$

We can justify this by noting that if $n \equiv 0 \pmod 3$, then there will be an equal number of upward- and downward-pointing triples, while if $n \equiv 1 \pmod 3$, then there will be one more upward-pointing triple than there are downward-pointing triples. The reason that $f$ is not defined for $n \equiv 2 \pmod 3$ is because this is never the number of unused circles in a row if we follow the above greedy algorithm.

To prove this, let $u_k$ be the number of unused circles when we get to row $k$. We note that $u_1 = 1$ and $u_n = n - f(u_{n-1})$. Since $u_n \pmod 3$ depends only on $n \pmod 3$ and $u_{n-1} \pmod 3$, we can simply step through values of $u_n$ until $n \pmod 3$ and $u_{n-1} \pmod 3$ repeat. We calculate:

| $n$ | $u_n$ | $n \pmod 3$ | $u_{n-1} \pmod 3$ |
|---|---|---|---|
| 1 | 1 | 1 | — |
| 2 | 0 | 2 | 1 |
| 3 | 3 | 0 | 0 |
| 4 | 1 | 1 | 0 |
| 5 | 3 | 2 | 1 |
| 6 | 3 | 0 | 0 |
| 7 | 4 | 1 | 0 |

Hence, we can see that there is a pattern with period 3, and hence $u_n$ will never be $2 \pmod 3$. To be more precise, we can derive all the $u_i$ values by induction. The base case is clear from the table above, and in the inductive step we calculate:

$$\begin{aligned} u_{6n} &= 3n \\ u_{6n+1} &= 3n+1 \\ u_{6n+2} &= 3n \\ u_{6n+3} &= 3n+3 \\ u_{6n+4} &= 3n+1 \\ u_{6n+5} &= 3n+3 \\ u_{6n+6} &= 3n+3, \end{aligned}$$

as desired. Hence, we also have $u_{2022} = 1011$, $u_{2023} = 1012$, and $u_{2024} = 1011$. At this point, we can stop using our greedy algorithm, and we can fill in the last three rows as follows: to fill in the unused 1011 circles in row 2024, we can use a $3 \times 1010$ parallelogram, leaving the circle in the 1014-th position unused. Then, we can fill in the remainder of the 2025-th and 2026-th rows using a $2 \times 1014$ parallelogram and a single upward-pointing triple.

**Comment.** When the greedy algorithm is applied, a pattern emerges which uses interlaced strips of 3-wide parallelograms with a series of upward-pointing triples nestled between them. It would be possible to use this pattern directly instead of generating it via the greedy algorithm.

# Problem 2. *(proposed by Daniel Naylor)*

ICMC is turning 9 years old! To celebrate, Dylan buys 9 candles to put on a birthday cake. He would like to place 8 of the candles in distinct positions so as to form two squares $ABCD$ and $EFGH$. Is it possible to do this so that, regardless of where he places the ninth candle $P$, it is true that

$$PA^2 + PB^2 + PC^2 + PD^2 = PE^2 + PF^2 + PG^2 + PH^2?$$

(Assume the birthday cake is the Euclidean plane, and each candle is a distinct point.)

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**Notes on Marking.** Most contestants who attempted the question took one of two main solution paths: a "constructive" or an "algebraic" one.

For a "constructive" proof, 3 marks were awarded for a clear and precise description *or* drawing (without justification!) of an arrangement of candles that guarantees the expression holds, 5 marks were then awarded for deducing *and* simplifying one side of the expression for the proposed arrangement, 1 mark was awarded for repeating this for the other side of the expression, and a final 1 mark was awarded for concluding their equality.

For the "algebraic" proof, 2 marks were awarded for *clearly* invoking some algebraic approach (e.g. law of cosines, British flag theorem were popular approaches) as the solution strategy, 6 marks could be obtained for then deducing a general expression with two variables (or equivalently, degrees of freedom) for the sum of squared distances from a square's vertices to a general point, and a final 2 marks were awarded for the conclusion that the obtained expression is satisfied by two congruent squares rotated around a shared centre.

Some contestants tried to prove that such an arrangement is not possible, but used arguments relevant to a correct solution. In such cases, relevant elements of their proof were marked independently of the incorrect conclusion.

A recurring pitfall that carried 0 marks was "simplifying" the target expression, dividing it through by $P$. Note that $PA^2$ means the distance from $P$ to $A$, squared.

In the future, when invoking a non-universally known theorem, contestants are heavily encouraged to explicitly state the main result of such a theorem, and clearly indicate where they use that result in their proof.

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## Solution 1. *(ICMC solution by Daniel Naylor)*

For a square $S$ in the plane, consisting of points $A, B, C, D$, let $f_S : \mathbb{R}^2 \to \mathbb{R}$ denote the function $f(P) = PA^2 + PB^2 + PC^2 + PD^2$. Then the question is asking us whether we can find squares $S_1, S_2$ with no overlapping points, such that the functions $f_{S_1}$, $f_{S_2}$ are the same.

For the square $S$ with coordinates $(1, 1), (1, -1), (-1, -1), (-1, 1)$, we have

$$\begin{aligned} f_S(x, y) &= ((x-1)^2 + (y-1)^2) \\ &\quad + ((x-1)^2 + (y+1)^2) \\ &\quad + ((x+1)^2 + (y+1)^2) \\ &\quad + ((x+1)^2 + (y-1)^2) \\ &= 4x^2 + 4y^2 + 8 \end{aligned}$$

By rotating and translating, we see that for any unit square $S$ with centre $O$, we have $f_S(P) = 4OP^2 + 8$. Thus if Dylan places $ABCD$ and $EFGH$ to form two unit squares with a common centre, then the desired equation is satisfied for all points $P$. One possible way of doing this is by using a regular octagon. $\square$

## Solution 2.

*(ICMC solution by Tony Wang)*

The equation of a general quadratic in the plane can be written as $f(P) = a \cdot PB^2 + c$, where $a$ and $c$ are real numbers, and $B$ is a point in the plane. If we write $P = (x, y)$ and $B = (x_0, y_0)$, then this can also be written as $f\Big((x, y)\Big) = a\Big((x - x_0)^2 + (y - y_0)^2\Big) + c$.

**Lemma:** The sum of two quadratics in the plane with leading coefficients $a_0$ and $a_1$ is another quadratic in the plane with leading coefficient $a_0 + a_1$.

*Proof.* We can compute the sum of two quadratics $a_0\Big((x - x_0)^2 + (y - y_0)^2\Big) + c_0$ and $a_1\Big((x - x_1)^2 + (y - y_1)^2\Big) + c_1$. This turns out to be

$$(a_0 + a_1)\left[\left(x - \frac{a_0x_0 + a_1x_1}{a_0 + a_1}\right)^2 + \left(y - \frac{a_0y_0 + a_1y_1}{a_0 + a_1}\right)^2\right] + c,$$

for some real constant $c$. We note that this is in the form of a general quadratic in the plane, and its leading coefficient is $a_0 + a_1$, as desired. $\square$

By the lemma, the expression $f(P) = PA^2 + PB^2 + PC^2 + PD^2$ will have the form $f(P) = 4 \cdot PX^2 + c$ for some point $X$. However, note that squares in the plane have four degrees of freedom (since squares can be uniquely determined by choosing two points to be its diameter, and each point has two degrees of freedom), and yet the equation has three degrees of freedom (two for $X$, and one for $c$). This implies that there must be two squares $ABCD$ and $EFGH$ which result in the same quadratic in the plane.

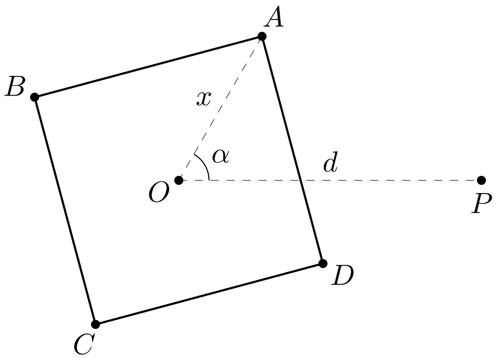
The only remaining issue is that these two squares may share a point. To prove this cannot be the case, note that if $X$ from the quadratic is not the centre of the square, then we can rotate the square 90° about its centre. The square has not changed position, but $X$ will have changed position, a contradiction as each square corresponds with exactly one quadratic in the plane. Hence, two squares that share the same quadratic must share the same centre. Finally, note that two squares that share the same centre cannot share any points unless they are the same square. Hence, we can pick two disjoint squares $ABCD$ and $EFGH$, and for those two squares, the expressions $PA^2 + PB^2 + PC^2 + PD^2$ and $PE^2 + PF^2 + PG^2 + PH^2$ will be identical. $\square$

**Comment.** It turns out that, given a square $ABCD$, we can write $PA^2 + PB^2 + PC^2 + PD^2$ as $4 \cdot PX^2 + c$, where $X$ is the centre of the square, and $c$ is twice the area of the square. This implies that two squares result in the same quadratic in the plane if and only if they are the same size and share the same centre – that is, if and only if one is a rotation of the other about its centre.

## Solution 3.

*(ICMC solution by Gergely Rozgonyi)*

We will show the result in the Comment for Solution 2. It will then follow, that any two squares that are the same size and have the same centre satisfy the relationship.




Using the notation in the figure above and a generic point $P$ that may be inside or outside the square, denote by $O$ the centre of the square $ABCD$, the lengths $d = OP$ and $x = OA$, and the angle $\widehat{POA} = \alpha$. Clearly, $x^2 = AB^2/2$.

By the law of cosines

$$\begin{aligned} PA^2 &= x^2 + d^2 - 2xd\cos(\alpha) \\ PB^2 &= x^2 + d^2 - 2xd\underbrace{\cos(90^\circ + \alpha)}_{-\sin\alpha} \\ PC^2 &= x^2 + d^2 - 2xd\underbrace{\cos(180^\circ + \alpha)}_{-\cos\alpha} \\ PD^2 &= x^2 + d^2 - 2xd\underbrace{\cos(270^\circ + \alpha)}_{\sin\alpha}. \end{aligned}$$

Summing the equations we have

$$PA^2 + PB^2 + PC^2 + PD^2 = 4x^2 + 4d^2 = 2 \cdot AB^2 + 4 \cdot OP^2,$$

independent of $\alpha$ and giving the required result. □

# Problem 3.

*(proposed by Andrija Živadinović)*

Let $a$, $b$, and $c$ be positive integers with $\gcd(a, b, c) = 1$ such that

$$2a^2 - b^2 - c^2 + 2bc - ab - ac = 0.$$

Show that $a$ is either an odd square number or two times an even square number.

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**Notes on Marking.** There were a lot of different correct ways to approach this problem, and there were quite a lot of unique and/or elegant solutions. Most of the solutions were generally similar to the Solution 2 below. In almost all of the solutions, the rearrangement of the given equation to $a(2a - b - c) = (b - c)^2$ was useful, and we awarded 1 mark for it, but only if it was explicitly written in that form, or any useful further ideas on how to use that form of the given equation were shown.

For Solution 1, we awarded 4 points for the first part, which was obtaining that the square-free part of $a$ was 1 or 2, 3 marks for case $k = 1$, and 4 marks for case $k = 2$, where both cases together carried 6 marks.

For Solution 2, showing that every odd prime has even exponent in the prime factorisation of $a$ was worth 4 points, with 1 mark deducted if it was proven that $\gcd(a, 2a - b - c)$ is a power of 2, but the comments about implications of this were missing. A few attempts involved proving that if an odd prime $p$ divides $a$, then also $p^2$ divides $a$, instead of fully showing that an exponent of $p$ in the prime factorisation of $a$ is even. We deducted 1 mark in such cases.

Resolving the case of even $a$ was worth 6 marks. A common mistake was that many solutions submitted missed the case of $v_2(a) = 1$ (i.e. $a \equiv 2 \mod 4$), where 1 mark was deducted if that case was easily solvable using only the arguments already presented, and 2 marks were deducted otherwise.

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## Solution 1.

*(ICMC solution by Tony Wang and Daniel Naylor)*

The equation can be rewritten in the form $a(2a - x) = y^2$, where $x = b + c$ and $y = b - c$. Write $a = kn^2$, where $k$ is square-free (that is, each prime factor divides $k$ at most once). We now have

$$k(2kn^2 - x) = \left(\frac{y}{n}\right)^2.$$

Since $k$ is square-free and yet $k(2kn^2) - x)$ is a square, each prime factor of $k$ must also appear in $2kn^2 - x$. From here, we deduce that $k \mid 2kn^2 - x$, or $k \mid x$. Each prime factor dividing $k$ hence also appears in $y$, and so we also have $k \mid y$. Now, we have $k \mid x + y = 2b$ and $k \mid x - y = 2c$. Since $k \mid a \mid 2a$ by definition, we must have $k \mid \gcd(2a, 2b, 2c) = 2$, as $\gcd(a, b, c) = 1$. So, $k = 1$ or 2. We now have two cases:

- Case 1: $k = 1$. If $2 \nmid y$, we must have $n \equiv 1 \pmod 2$. If $2 \mid y$, so $2 \mid x$, we write $x = 2x_1$, $y = 2y_1$, so the last equation becomes $n^2 - x_1 = 2\left(\frac{y_1}{n}\right)^2$. If we had $2 \mid n$, then we must have $2 \mid x_1$, but as $n \mid y_1$, we also have $2 \mid y_1$, so $4 \mid x$, and $4 \mid y$. Then $2 \mid b = \frac{x+y}{2}$ and $2 \mid c = \frac{x-y}{2}$, but since we also have $2 \mid n \mid a$, we get $\gcd(a, b, c) \geq 2$, a contradiction. Thus, it must be $n \equiv 1 \pmod 2$.

- Case 2: $k = 2$. We then have $2(4n^2 - x) = (\frac{y}{n})^2$, so $2 \mid y$, and hence $2 \mid x$. Writing $x = 2x_1$ and $y = 2y_2$, we get $2n^2 - x_1 = (\frac{y_1}{n})^2$. Since $2 \mid a$, $\gcd(a, b, c) = 1$, and $b \equiv c \pmod 2$ (as $2 \mid x$, $2 \mid y$), we must have $b \equiv c \equiv 1 \pmod 2$. If $2 \nmid n$, then, by the above, $2 \mid x_1$ if and only if $2 \mid y_1$. Since $b \equiv c \equiv 1 \pmod 2$, at least one of $x_1$ and $y_1$ must be even (if $b \equiv c \pmod 4$, that is $y_1$, otherwise that is $x_1$), so by the above, both $x_1$ and $y_1$ are even. We then have $2 \mid b = x_1 + y_1$ and $2 \mid c = x_1 - y_1$, giving us a contradiction, as $2 \mid a$ and $\gcd(a, b, c) = 1$. Thus, it must be the case that $2 \mid n$.

Hence, $a$ must be either an odd square number or two times an even square number. □

**Comment.** Continuing from here, we can parametrise all solutions: the general solutions for $(a, b, c)$ are, for all $k \in \mathbb{N}$:

$$\left((2k-1)^2,\ (2k-1)^2 - \frac{d^2 - (2k-1)d}{2},\ (2k-1)^2 + \frac{d^2 - (2k-1)d}{2}\right)$$

for $|d| \leq 2k - 1$ and $\gcd(d, 2k - 1) = 1$;

$$\left(2 \cdot (2k)^2,\ 2 \cdot (2k)^2 - d^2 + 2kd,\ 2 \cdot (2k)^2 - d^2 - 2kd\right),$$

for $|d| \leq 2k$ and $\gcd(d, 2k) = 1$.

## Solution 2. *(solution by contestants)*

Rearranging the given equation, we get $a(2a - b - c) = (b - c)^2$. If $p$ is an odd prime, we have that if $p \mid a$, we must also have $p \mid (b - c)^2$, so $p \mid (b - c)$. If further $p \mid (2a - b - c)$, we get $p \mid (b + c)$, so $p \mid 2b$ and $p \mid 2c$. Since $p$ is an odd prime, $p \mid b$ and $p \mid c$, contradicting $\gcd(a, b, c) = 1$. Thus, we cannot have $p \mid (2a - b - c)$. If $b = c$, we must have $2a = b + c$, so $a = b = c$, implying $a = 1$, so $a$ is an odd square. Otherwise, if we let $v_p(b - c) = z$ (the largest power of $p$ dividing $(b - c)$), we have $v_p(a) = 2z$, which is an even number. In particular, if $a$ is odd, it must be a square.

If $a$ is even, let $v_2(a) = k > 0$ (the largest power of 2 dividing $a$). As $2 \mid a \mid (b - c)^2$, we have $2 \mid (b - c)$, so $\gcd(a, b, c) = 1$ gives us that both $b$ and $c$ are odd. Then since $2c \equiv 2 \mod 4$, and $b + c = (b - c) + 2c$, exactly one of the numbers $(b - c)$ and $(b + c)$ is divisible by 4, and the other one is congruent to $2 \mod 4$.

If $4 \mid (b + c)$, then $v_2((b - c)^2) = 2 = v_2(a) + v_2(2a - b - c)$. However, $v_2(a) \geq 1$, and $v_2(2a - b - c) \geq 2$, giving us a contradiction. Hence, we must have $4 \mid (b - c)$, and $v_2(2a - b - c) = 1$. Thus, if we let $v_2(b - c) = l \geq 2$, we get that $v_2(a) = 2v_2(b - c) - v_2(2a - b - c) = 2l - 1$, which is an odd number greater than 1. Together with the above result, this shows that $a$ is twice an even square, as required.

**Comment.** If we let $\gcd(a, 2a - b - c) = 2^r$ for $r \geq 0$ and write $a = 2^r m^2$, $2a - b - c = 2^r n^2$ and $b - c = 2^r mn$ with $\gcd(m, n) = 1$, we can solve for $b$ and $c$, and see that the given condition $\gcd(a, b, c) = 1$ forces $r = 0$ or $r = 1$. This quickly gives us the general solution form for $(a, b, c)$ stated in the comment of the first solution above.

## Solution 3. *(solution by Matteo Damiano of SNS, Pisa)*

We will use the well-known fact that if $A, B, C, D$ are positive integers with $AB = CD$, then there exist positive integers $P, Q, R, S$ such that $A = PQ$, $B = RS$, $C = PR$, and $D = QS$.

We rearrange the given equation as $a(2a - b - c) = (b - c)^2$, and note that if $b = c$, we must have $2a = b + c$, so $a = b = c$, and the given condition $\gcd(a, b, c) = 1$ gives us $a = 1$, so $a$ is an odd square. From now on, we assume without loss of generality that $b > c$. Using the fact stated above, we can find positive integers $\alpha, \beta, \gamma, \delta$ such that $a = \alpha\beta$, $2a - b - c = \gamma\delta$, $b - c = \alpha\gamma$, $b - c = \beta\delta$.

Since now $\alpha\gamma = \beta\delta$, using the fact stated above again, we can find positive integers $u, v, w, z$ such that $\alpha = uv$, $\gamma = wz$, $\beta = uw$, $\delta = vz$. Solving the system we get in terms of $u, v, w, z$, we get $a = u^2vw$, $2b = vw(2u - z)(u + z)$, $2c = vw(2u + z)(u - z)$. Thus, $\gcd(a, b, c) = 1$ condition gives us $vw \in \{1, 2\}$.

If $vw = 1$, we have $a = u^2$, $2b = (2u - z)(u + z)$, $2c = (2u + z)(u - z)$. If $u$ was even, we would have $2b \equiv -z^2 \equiv z^2 \equiv 0 \mod 2$, so $z$ is even. Then $4 \mid 2b, 2c$, so $2 \mid a, b, c$, giving us a contradiction with $\gcd(a, b, c) = 1$. Hence, $u$ must be odd in this case.

If $vw = 2$, we have $a = 2u^2$, $b = (2u - z)(u + z)$, $c = (2u + z)(u - z)$. If $u$ was odd, we would have $b \equiv z(z+1) \equiv 0 \mod 2$, and similarly $c \equiv 0 \mod 2$. Again, we would get $2 \mid a, b, c$, contradicting $\gcd(a, b, c) = 1$. Hence, $u$ must be even in this case.

Thus, $a$ is either an odd square, or twice an even square.

# Problem 4. *(proposed by Dylan Toh and Tony Wang)*

Daniel and Andrija play a game in the Euclidean plane. Daniel chooses a set of 2025 distinct points such that no three are collinear. Andrija then draws $m$ lines in the plane, none of which pass through any of Daniel's points. Andrija's lines split the plane into regions, and he wins if each region contains at most 1 point. Find the smallest $m$ such that Andrija has a winning strategy regardless of Daniel's choice of points.

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**Notes on Marking.** Working out the right answer on this problem felt like non-trivial progress towards the answer, so we awarded 1 mark for any attempt that mentioned 1013 or a corresponding generalisation to when we have a number of points different to 2025. The lower bound proof is a bit simpler than the upper bound, so we allocated 4 marks for the lower bound and 5 marks for the upper bound.

For the lower bound, 1 mark was given for putting the points on a convex shape, and the last 3 were awarded for completing the proof of the lower bound. Many contestants attempted to prove the lower bound using an inductive approach, arguing that the number of point-containing regions increases by at most 2 each time a line is added. However, using two horizontal lines and then adding one vertical line shows that this is false. Hence, an edge-cutting argument as in Solution 1 needed to be used to receive the 3 marks.

For the upper bound, there were quite a few different solutions. The most common is the one shown in Solution 1. In this construction, after drawing a line dividing the points into sets of 1012 and 1013 (the existence of which we granted without the need for discrete intermediate value theorem), many contestants tried to draw a line separating the two points on either side of the diving line whose projections to it were the most extreme (for example, the rightmost). However, this approach doesn't work, and this can be seen more easily if the most extremal projections from the two sides differ greatly. Another pitfall for some contestants was not taking enough precaution in their construction that their other lines not pass through a point, or in some cases, that the point about which they rotated a line was not collinear with two of Daniel's points.

Many candidates had proofs of weaker upper and lower bounds, but unfortunately the vast majority of these shared no key ideas with the solutions below, so we didn't award any of the above partial marks except in very few cases. That is, an answer or bound of 1013 lines was needed to earn most partial marks associated with the lower and upper bound.

Finally, many contestants assumed that Andrija is a female name, and therefore misunderstood the problem. In fact, it is South Slavic male name, cognate with the name "Andrew". We realise that perhaps this was not as clear as it could have been. A few contestants neglected the condition that no three points can be on a line. In this case, the answer would indeed be $m = 2024$, after Daniel draws all his points on a single line. However, no marks were awarded for this.

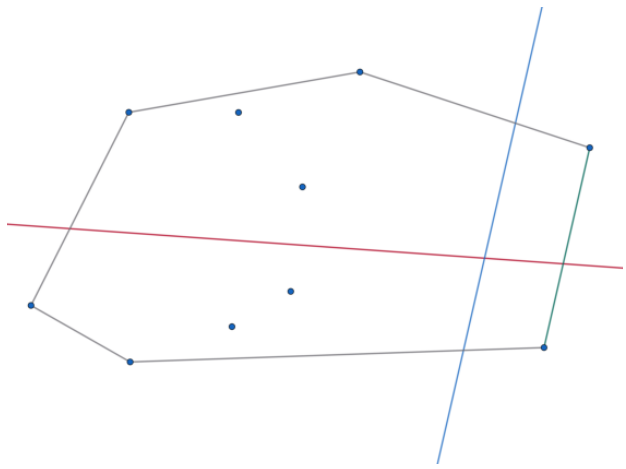
---

## Solution 1. *(ICMC solution by Andrija Živadinović)*

We will prove that, in general, for $n \in \mathbb{N}$, given $2n-1$ or $2n$ points in a plane such that no three are collinear we need $n$ lines to guarantee we can split the plane in regions using them, such that there is at most 1 point in each region.

To see that $n-1$ line is not enough, consider $2n-1$ vertices of a regular $(2n-1)$-gon in the plane (and if we have $2n$ points, put the last one anywhere in the plane, satisfying the problem conditions). To split two adjacent vertices of this polygon into separate regions, we need to have at least one of the lines intersecting the side of this polygon connecting them. As any line not containing any of the vertices of this polygon can intersect at most 2 different sides of this polygon, using $n-1$ line we can intersect at most $2n-2 < 2n-1$ sides of this polygon, so at least one pair of adjacent vertices will not be split into two separate regions. Thus, $n-1$ line is not enough.

Now we prove that $n$ lines is enough. Since the problem is strictly harder for $2n$ points than for $2n-1$ points, assume we are given $2n$ points in the plane, no three of which are collinear. First, we can choose an arbitrary line not parallel to any line joining two of the points, and then translate the this line until it is positioned such that there are exactly $n$ points on each side of it (red line in the picture below). This is possible by the discrete intermediate value theorem. We then repeat the following process: we draw a line that splits exactly one point on each side of the red line from the rest of the currently unsplit points. After $n-1$ iterations of this, all $2n$ points will be split into separate regions using exactly $n$ lines. Hence, it remains to prove that such a line can be found.



To prove this, consider the convex hull of all the currently unsplit points. As there is at least one such point on each side of the red line, there must be at least one side of this convex hull intersecting the red line. The line determined by that side contains exactly one of the currently unsplit points on each side of the red line, and all the other currently unsplit points are on the same side of it. Thus, translating this line a little towards the rest of the currently unsplit points without changing its direction will give us a line with desired property. Hence, we are done.

Since $n = 2025$ given in the problem, the smallest $m$ for which Andrija has a winning strategy regardless of Daniel's choice of points is $m = 1013$. □

**Comment.** The proof of the lower bound also works if we put the points on any convex curve.

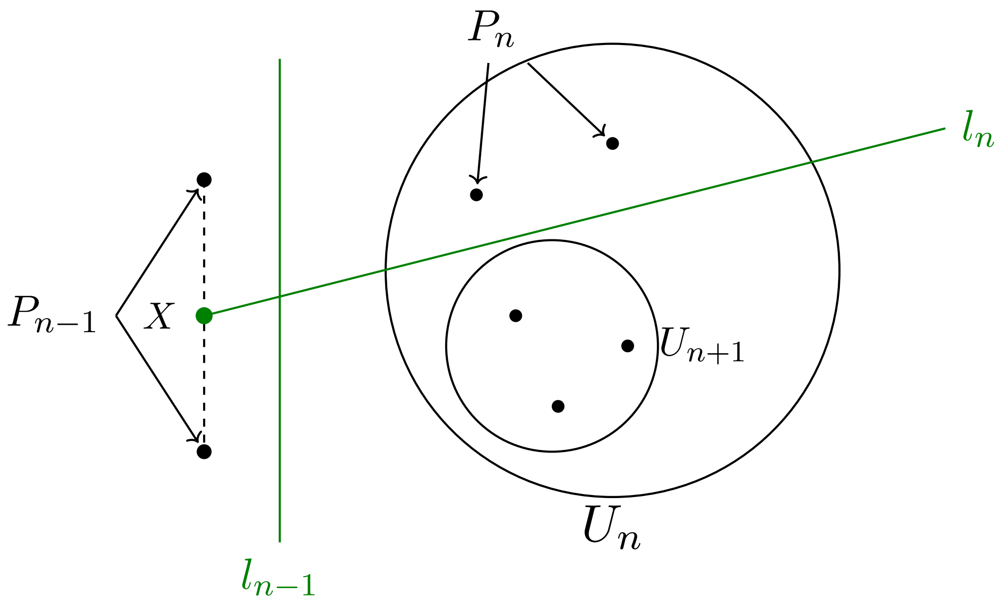
## Solution 2. *(solution by Contestants)*

We give another proof of the upper bound: we will show that given any set $S$ of 2025 points, it is possible to find 1013 lines that divide the plane into regions such that no region has more than 1 point.

The strategy will be the following: the first line will be used to separate a pair of points $P_1$ from the rest of $S$. For $2 \leq n \leq 1012$, the $n$-th line will be used to separate a pair of points $P_n$ from the current collection of unseparated points (the "unseparated points" is the set $U_n = S \setminus (P_1 \cup P_2 \cup \cdots \cup P_{n-1})$). The $n$-th line will also split the points in $P_{n-1}$. Finally, the 1013-th line will be used to split the pair of points $P_{1012}$.

For the first line, we can take a side of the convex hull, but translated inwards a little bit. The 1013-th line is very easy, so we now just focus on showing that the task at the $n$-th stage is possible to do.

For the $n$-th stage, let $X$ be any point on the line segment connecting $P_{n-1}$ such that $X$ does not lie on a line connecting any of the pairs of points of $S$. Now consider a rotating line $l$ that rotates about $X$, and starts as passing through the pair of points $P_{n-1}$. At some point, $l$ will separate $U_n$ into two pieces, with one piece having size 2 (here, we use the choice of point $X$ to ensure that as we rotate the line, the line passes at most one point at a time). We then take such an $l$ as our choice for the $n$-th line.




□

## Solution 3. *(solution by Contestants)*

We give another proof of the upper bound. Again, let $S$ be the given set of 2025 points, and start by picking a line that divides $S$ into pieces of sizes 1013 and 1012, as in Solution 1. Call the sides $A$ and $B$.

We now draw another 1012 lines, and with each line we aim to separate some pair points that share a region of $A$ and some pair of points that share a region of $B$. It could be the case that after $n$ lines we have already separated all points in $A$; if this is the case then with the $(n+1)$-th line we only aim to separate some points that share a region of $B$. Similarly for the other way round.

It now suffices to show that if we have a region $R_A$ containing at least 2 points of $A$, and a region $R_B$ containing at least 2 points of $B$, then it is possible to construct a line that separates a pair of points in $R_A$ and separates a pair of points in $R_B$. Let $a_1, a_2$ be a pair of points in $R_A$ and $b_1, b_2$ a pair of points in $R_B$. Let $a$ be any point on the line segment $a_1a_2$ that does not lie on line $b_1b_2$. Then let $b$ be any point on line segment $b_1b_2$ such that $ab$ doesn't pass through any point of $S$. Then line $ab$ works. □

# Problem 5. *(proposed by Tony Wang)*

Does there exist a twice-differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f''(x) > f(x) > f'(x) > 0 \qquad \text{for all } x \in \mathbb{R}?$$

---

**Notes on Marking.** The common theme among all solutions to this problem is that at some point they study the behaviour as $x \to -\infty$; see the comment at the end of Solution 2 for an example that demonstrates that this is necessary. Furthermore, all of the solutions below consider an auxiliary function (a function that is not one of $f$, $f'$ or $f''$) and consider its derivative in some way.

For the above reasons, we awarded 1 mark for considering a useful auxiliary function and 2 marks for proving the existence of the limit of some relevant function (auxiliary or not) as $x \to -\infty$. We also awarded a further 2 marks for any attempt that ruled out a case for the value of a limit (for example, showing that it must be equal to 0).

For attempts that would have otherwise received full marks, we deducted 1 mark if they claimed $f''(x) \to 0$ as $x \to -\infty$ (when the claim should have instead been that $\liminf_{x\to-\infty} f''(x) = 0$), and we deducted up to 2 marks on attempts that involved integrating $f''$, depending on how difficult the fix was.

Common mistakes included variants of the following:

- Incorrectly claiming that if $g(x) > 0$ for all $x$, then $\int_0^x g(t)\,\mathrm{d}t > 0$ for all $x$ (note that actually the lower bound on this integral only follows for $x > 0$). This kind of mistake was very costly because, as mentioned above, it is impossible to solve this question without considering negative values of $x$.

- Proving inequalities $a > c$, $b > d$ and then incorrectly deducing $a - b > c - d$. Usually the attempt was written in a way that made it less obvious that it was relying on this type of incorrect reasoning.

- Forgetting $+C$ when doing an indefinite integral; one easy way to avoid this kind of mistake is to use definite integrals instead.

- Claiming that if $g(x) > h(x)$ for all $x$, then $g'(x) > h'(x)$.

- Claiming that if $g$ is differentiable, then $\int_a^b g'(t)\mathrm{d}t = g(b) - g(a)$. The Fundamental Theorem of Calculus says that this *is* true if $g'(t)$ is continuous, but if we don't know that $g'(t)$ is continuous, then it can be the case that this integral does not even exist! This applied to some attempts that involved integrating expressions involving $f''$, though these were generally fixable by an application of the Mean Value Theorem (though sometimes it had to be applied to an auxiliary function).

---

## Solution 1. *(ICMC solution by Tony Wang)*

We show that the answer is no. First, note that since $f$ is increasing and strictly positive, it must to converge to some value $c \geq 0$ as $x \to -\infty$. We now split into two cases depending on the value of $c$:

- If $c = 0$, then there exists a $d < 0$ sufficiently small such that $f(d) < f(0) - f'(0)$. We also have $\frac{\mathrm{d}}{\mathrm{d}x}\Big(f(x) - f'(x)\Big) = f'(x) - f''(x) < 0$, so $f(x) - f'(x)$ is decreasing. Putting these together yields $f(d) - f'(d) > f(0) - f'(0) \implies 0 > f'(d)$, a contradiction.
- If $c > 0$, then $f''(x) > c$ for all real numbers $x$. Hence, $f'(x)$ increases with gradient at least $c$ everywhere, so $f'(x) \to -\infty$ as $x \to -\infty$. □

## Solution 2. *(ICMC solution by Daniel Naylor)*

Since $f(x) > 0$, we can perform the substitution $f(x) = e^{g(x)}$. The given inequality then transforms to

$$(g'(x)^2 + g''(x))e^{g(x)} > e^{g(x)} > g'(x)e^{g(x)} > 0.$$

Dividing by $e^{g(x)}$ (which is necessarily positive), we get

$$g'(x)^2 + g''(x) > 1 > g'(x) > 0.$$

Now let $h(x) = g'(x)$, so

$$h(x)^2 + h'(x) > 1 > h(x) > 0.$$

A consequence of this is that $h'(x) > 1 - h(x)^2 > 1 - 1^2 > 0$. Hence for $x < 0$, we have $h(x) < h(0)$, and so for $x < 0$ we have

$$h'(x) > 1 - h(x)^2 > 1 - h(0)^2 > 0.$$

Hence, by the Mean Value Theorem,

$$h(x) < h(0) + x(1 - h(0)^2),$$

so if $x \ll 0$, we have $h(x) < 0$, a contradiction. □

**Comment.** If we take $h(x) = 1 - e^{-2x}$, then for $x > 0$, we have $0 < h(x) < 1$ and

$$h'(x) = 2e^{-2x} > 2e^{-2x} - e^{-2x} = 1 - h(x)^2.$$

Hence this choice of $h(x)$ satisfies

$$h(x)^2 + h'(x) > 1 > h(x) > 0$$

for $x > 0$. By following the earlier reasoning backwards, we get that the function $f(x) = e^{\int_0^x h(t)\,\mathrm{d}t} = e^{x + \frac{1}{2}(e^{-2x} - 1)}$ satisfies $f''(x) > f(x) > f'(x) > 0$ for all $x \in \mathbb{R}_{>0}$.

The existence of such a function shows that any attempt to derive a contradiction in this problem must consider negative values of $x$. In fact, by horizontal translation invariance, we can further deduce that any attempt to derive a contradiction must consider behaviour as $x \to -\infty$. In particular, it is not possible to get a contradiction in this problem by just considering what happens as $x \to \infty$, or by just considering what happens in some finite interval.

## Solution 3. *(solution by Contestants)*

Since $f(x) > f'(x) > 0$ for all $x$, we have $0 < \frac{f'(x)}{f(x)} < 1$. Note that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f'(x)}{f(x)}\right) = \frac{f''(x)}{f(x)} - \left(\frac{f'(x)}{f(x)}\right)^2 > 0,$$

so $\frac{f'(x)}{f(x)}$ must be strictly increasing. As $x \to -\infty$ we get $\frac{f'(x)}{f(x)} \to C < 1$, where $C \neq 1$ as it is strictly increasing. Hence,

$$\lim_{x \to -\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left(f'(x)/f(x)\right) \geq 1 - C^2 > 0$$

contradicting the convergence of $f'/f$ as $x \to -\infty$ □

**Comment.** Note that the function $h(x)$ in Solution 2 is equal to $\frac{f'(x)}{f(x)}$, which is why these two solutions look a bit similar.

## Solution 4. *(solution by Kevin Barreto of Cambridge)*

We shall show there is no such $f$. Suppose, for the sake of contradiction, there was such a function $f$. Since $f(x) > 0$ for all $x \in \mathbb{R}$, we may divide through by it to obtain the inequality

$$1 > u(x) > 0 \qquad \text{where} \qquad u(x) := \frac{f'(x)}{f(x)}.$$

Now, by the quotient rule,

$$u'(x) = \frac{f''(x)f(x) - (f'(x))^2}{(f(x))^2} = \frac{f''(x)}{f(x)} - \left(\frac{f'(x)}{f(x)}\right)^2 = \frac{f''(x)}{f(x)} - (u(x))^2.$$

Now, by the first given inequality, we have $\frac{f''(x)}{f(x)} > 1$, i.e.

$$u'(x) > 1 - (u(x))^2 \quad \Longleftrightarrow \quad \frac{u'(x)}{1 - (u(x))^2} > 1$$

as $u(x) \in (0,1)$. Now for a $y < 0$, by Mean Value Theorem (since $(\text{artanh}(u(x)))' = \frac{u'(x)}{1-(u(x))^2} > 1$) we have

$$\text{artanh}(u(0)) - \text{artanh}(u(y)) > -y,$$

i.e.

$$0 < \text{artanh}(u(y)) < \text{artanh}(u(0)) + y \qquad \forall y \in \mathbb{R},$$

where the first inequality comes from noting that $\text{artanh}(t) > 0$ for $t > 0$, and recalling that $u(y) \in (0,1)$ for all $y \in \mathbb{R}$. Sending $y \to -\infty$, the right-hand side of the above inequality goes to $-\infty$, whereas the left-hand side remains positive. This gives a contradiction, so there does not exist such an $f$. $\square$

# Problem 6.

*(proposed by Tony Wang)*

Ishan has a fair coin with an equal chance of landing on heads ($H$) or tails ($T$), and would like to simulate a fair 5-sided die using a *correspondence.* A correspondence assigns which (possibly infinite) set of finite sequences of coin tosses correspond to each face of the die. Using a correspondence, Ishan will toss the coin until his sequence of coin tosses matches one assigned to a face $f$ of the die exactly (i.e. not just as a subsequence), at which point Ishan stops tossing the coin and declares the result of the simulated die-roll to be $f$. In order for the correspondence to be well-defined, no sequence can be assigned to more than one face, and no assigned sequence may be the start of any other assigned sequence. For example, if the sequences $H$, $TT$, $THT$, and $THH$ are assigned to faces of the die, then no other sequences may be assigned.

Over all possible correspondences, what is the smallest expected number of times Ishan will need to toss the coin in order to simulate a fair 5-sided die?

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**Notes on Marking.** The mark scheme awarded 1 mark for describing an explicit optimal correspondence, 1 mark for writing down an answer of 3.6, and 8 marks for the proof of optimality.

- All optimal correspondences are essentially isomorphic to the correspondence presented in Solution 1 below, and hence other correspondences do not achieve this mark. Naive recursions – that is, assigning as many as possible at a given level and starting over if an unassigned sequence at that level was reached – usually achieved an expected value of 4.8 or $4.2666\ldots = 64/15$ for level 3 and level 4 naive recursions, respectively. A promising correspondence is to interpret the coin tosses as generating a random binary number between 0 and 1, and to stop tossing when it was certain the binary number would fall into one of the bins $\left[\frac{i-1}{5}, \frac{i}{5}\right)$ for $i = 1, \ldots, 5$. Unfortunately, this achieves an expected value of 4.

- The answer must have been written in short form (that is, as a numerical expression with finitely many terms) in order to achieve this mark. Many contestants wrote a correct infinite series but made a mistake while evaluating it, or were unable to evaluate it – we did not award the answer mark in these cases.

- Common deductions for the proof of optimality were made when contestants worked mainly with the lengths of sequences assigned to each face instead of the sequences themselves. In some of these cases, 1 mark was deducted for not showing that five sets of sequence lengths each with associated probability 0.2 could be turned back into a correspondence (see the Claim in Solution 2). Care was also needed to show that a $2\frac{k}{2^k}$ term could be turned into the smaller $\frac{k-1}{2^{k-1}}$ term. Describing a correspondence in terms of just the sequence lengths was a common way to lose a mark as well.

Many contestants also did not interpret this problem correctly, or did not provide a correspondence which simulated a *fair* die. (To clarify, an outside observer should not be able to tell based off the results alone whether Ishan is equipped with a real 5-sided die or with just a coin.) In these cases usually a mark of 0 was awarded, unless there were ideas useful to the problem.

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## Solution 1.

*(ICMC solution by Tony Wang)*

We will show that the answer is 3.6. We will model each correspondence as a rooted binary tree, with leaf nodes corresponding to assigned sequences. Call a correspondence that achieves the smallest expected coin tosses *optimal*.

**Lemma 1:** An optimal correspondence cannot have two leaves at the same level assigned to the same face of the die.

> *Proof.* Suppose for the sake of contradiction that an optimal correspondence $\mathcal{C}$ has two leaves $V_1$ and $V_2$ at the same level assigned to the same face. If those two leaves are not children of the same parent vertex, then swap the sibling of $V_1$ (which is not necessarily a leaf) with $V_2$ in the tree. This doesn't change the probabilities or the expected value of the correspondence, since it doesn't change the lengths of the involved assigned sequences.
>
> Next, combine $V_1$ and $V_2$ into their shared parent vertex, so that their parent vertex is now assigned to the same face $V_1$ and $V_2$ were assigned to. This doesn't change the probabilities of the correspondence, and it strictly decreases the expected value. That is, the result is still a valid correspondence, but it has a lower expected value of coin tosses. Hence, the correspondence $\mathcal{C}$ is not optimal, a contradiction. □

**Lemma 2:** The optimal correspondence assigns a single sequence of each length in the set $S = \{3, 4, 7, 8, 11, 12, \ldots\}$ to each face of the die.

> *Proof.* As the sum of reciprocals of powers of two must be equal to 0.2, and because each power of two can only be used once as per Lemma 1, we need to find a set of exponents $S \in \mathbb{Z}^+$ such that
>
> $$0.2 = \sum_{n\in S} 2^{-n}.$$
>
> However, there is only one way to do this, which is clear once we consider that the binary representation of 0.2 is $0.\overline{0011}_2$. Note that this also corresponds to the set $\{3, 4, 7, 8, 11, 12, \ldots\}$, as desired. □

We will now construct a correspondence with one leaf assigned to each face at every level in $S$:

- At the root level, there is one unassigned vertex.
- At level 1, there are two unassigned vertices.
- At level 2, there are four unassigned vertices.
- At level 3, five of the eight leaves are assigned, one to each face, leaving three unassigned vertices.
- At level 4, five of the six leaves are assigned, one to each face, leaving one unassigned vertex, and allowing us to repeat this construction from this vertex as if it were the root of the tree.

Since this is a recursive correspondence, we can see that there will be one sequence assigned to each face at the levels $4k + 3$ and $4k + 4$, which corresponds exactly to our set above. The total probability assigned to each face will be equal by symmetry, this guarantees that the correspondence simulates a fair die. Finally, the expected value $E$ of coin tosses satisfies the equation

$$E = 3 \cdot \frac{5}{8} + 4 \cdot \frac{5}{16} + (E + 4) \cdot \frac{1}{16},$$

and so we calculate that $E = 3.6$. □

## Solution 2. *(solution by Massimiliano Foschi of SNS, Pisa)*

We use the same correspondence as above which we know simulates a fair die and achieves a expected value of $\frac{18}{5}$, so it just remains to prove optimality. Letting $T$ be a random variable representing the number of tosses required to reach an assigned sequence, the expected value is

$$\sum_{n=1}^{\infty} n \cdot P(T=n) \;=\; \sum_{n=1}^{\infty}\sum_{i=1}^{n} P(T=n) \;=\; \sum_{i=1}^{\infty}\sum_{n=i}^{\infty} P(T=n) \;=\; \sum_{k=0}^{\infty} P(T>k),$$

where we can verify the second equality by counting the number of occurrences of $P(T=n)$ for each integer $n$.[†] This means that, we can bound the expected value from below if we can minimise each term $P(T>k)$ simultaneously.

Note that, amongst the $2^k$ sequences of length $k$, at most $\frac{2^k}{5}$ can be assigned to each face. However, since an integer number must be assigned, the upper bound is in fact $\left\lfloor \frac{2^k}{5} \right\rfloor$, leaving $2^k \pmod 5$ unassigned. Hence, we can inductively show over the non-negative integers $m$ that

- $P(T>4m) = \frac{1}{2^{4m}}$,
- $P(T>4m+1) = \frac{2}{2^{4m+1}}$,
- $P(T>4m+2) = \frac{4}{2^{4m+2}}$, and
- $P(T>4m+3) = \frac{3}{2^{4m+3}}$.

We can now calculate the minimum expected value to be

$$\begin{aligned}
\sum_{k=0}^{\infty} P(T>k) &\geq \sum_{m\geq 0} P(T>4m) + P(T>4m+1) + P(T>4m+2) + P(T>4m+3) \\
&= \sum_{m\geq 0} \frac{1}{2^{4m}} \left( \frac{1}{1} + \frac{2}{2} + \frac{4}{4} + \frac{3}{8} \right) \\
&= \frac{16}{15} \cdot \frac{27}{8} \\
&= \frac{18}{5},
\end{aligned}$$

and hence we are done.

## Solution 3. *(ICMC solution by Daniel Naylor)*

**Claim:** A correspondence naturally induces a family $\mathcal{S} = \{S_1, S_2, S_3, S_4, S_5\}$ with each $S_i$ being a multi-set of positive integers and having $\sum_{x \in S_i} 2^{-x} = \frac{1}{5}$, and moreover every such family induces a correspondence.

The important feature of this claim is the 'moreover' part at the end of it: in other words, what this claim is saying is that the subtle conditions about "no assigned sequence may be the start of any other assigned sequence" are in fact not hard to satisfy once you have found suitable elements for your sets $S_i$.

[†] More intuitively, we are rewriting the expected value as the sum of a triangular array of $P(T=n)$, where the usual definition of expected value takes the sum of the row sums, but the right-hand side is the sum of the column sums. This interpretation is also used in Solution 3.

*Proof.* The first part is clear, so we only prove the 'moreover' part. Given such a family $\mathcal{S}$, we will iteratively construct a rooted binary tree, with each leaf being an element of a set $S_i$. To do this, we start with all the 1s in $S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$, then all the 2s, then all the 3s, and so on. When we move on to a new number, we extend all unused leaves of the tree down. This process can continue forever (or as long as needed) precisely because of the assumed condition on the sums of $2^{-x}$ terms. □

We now want to minimise the expected number of coin flips over all families $\mathcal{S} = \{S_1, S_2, S_3, S_4, S_5\}$ as described in the claim. To do this, note that

$$\text{expected number of coin flips} = \sum_{i=1}^{5} \sum_{x \in S_i} \frac{x}{2^x},$$

so it suffices to find an optimal set $S_1$, as we can then take $S_i = S_1$ for the other $i$. By 'optimal', we mean a set $S_1$ that minimises $\sum_{x \in S_1} \frac{x}{2^x}$.

Note that $\sum_{x \in S_1} \frac{x}{2^x}$ equals

$$\sum_{x \in S_1} \frac{x}{2^x} = \sum_{x \in S_1} \sum_{\substack{n \in \mathbb{N} \\ x \geq n}} 2^{-x} = \sum_{n \in \mathbb{N}} \sum_{\substack{x \in S_1 \\ x \geq n}} 2^{-x}. \tag{$*$}$$

Now note that

$$\sum_{\substack{x \in S_1 \\ x \geq n}} 2^{-x} = \frac{1}{5} - \sum_{\substack{x \in S_1 \\ x < n}} 2^{-x},$$

and note that for each fixed $n$, the left hand side achieves its minimum when $S_1$ is the set induced by the binary expansion of $\frac{1}{5}$. Thus, if we take $S_1$ to be the set corresponding to the binary expansion of $\frac{1}{5}$, then $S_1$ pointwise minimises the sum on the right hand side of ($*$), and hence this choice of $S_1$ minimises the desired quantity.

Calculating the expected number of flips in this case is left as an exercise for the interested reader. □

**Comment.** The same solution works with 5 replaced by any number, i.e. considering binary expansions always gives an optimal strategy. The only part that changes is calculating the expected number of flips using the binary expansion.