IMPERIAL-CAMBRIDGE MATHEMATICS COMPETITION

9th Edition (2025–2026)

ROUND TWO

Official Solutions (Provisional)*

Last updated: 9 Mar 2026, 6am UTC Time

*A solution may receive full or partial marks even if it does not appear in this booklet.

Problem 1. *(proposed by Tony Wang)*

Prove that for all odd-degree polynomials p, there exist real numbers a and c and a non-zero real number d such that $f(a-d)+c = f(a) = f(a+d)-c$.

Notes on Marking. None yet.

Solution 1. *(solution by ICMC Committee)*

Note that the problem is trivial if f has degree 1. So assume from now on that the degree of f is greater than 2.

We claim that there is always a solution with $d = 1$. This is equivalent to saying that there exists a real number a such that $f(a-1) - f(a) = f(a+1) - f(a)$, since we can then set c to be this value.

Let $g(x) = f(x-1) + f(x+1) - 2f(x)$. By the above, it suffices to show that g has a root. To do this, we will show that g is an odd degree polynomial.

Suppose that $f(x)$ is of odd degree n ;and that

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \text{lower order terms.}$$

Then:

- The coefficient of x^n in $g(x)$ is $a_n + a_n - 2a_n = 0$.
- The coefficient of x^{n-1} in $g(x)$ is $(-na_n + a_{n-1}) + (na_n + a_{n-1}) - 2a_{n-1} = 0$.
- The coefficient of x^{n-2} in $g(x)$ is

$$\left(\binom{n}{2}a_n - (n-1)a_{n-1} + a_{n-2}\right) + \left(\binom{n}{2}a_n + (n-1)a_{n-1} + a_{n-2}\right) - 2a_{n-2} = 2\binom{n}{2}a_n \neq 0.$$

Thus g is on odd-degree polynomial, as desired. □

Problem 2. *(proposed by Dylan Toh)*

On the Euclidean plane, there are 11 bells of distinct tones and 2026 stationary villagers. All 11 bells are simultaneously rung exactly once. Supposing that sound travels at a constant speed, and that no villager hears two or more bells at the same time, prove that there are two villagers who hear the same order of bell tones regardless of where the bells and the villagers are.

Notes on Marking. None yet.

Solution 1. *(solution by ICMC Committee)*

A pair of villagers hear a different order if and only if they hear some pair in a different order. For any pair of bells, the order in which they are heard by a villager depends precisely on which side of the perpendicular bisector the villager is situated. Combining these two observations, we have that a pair of villagers hear a different order if and only if there is a pair of bells b_1, b_2 such that the villagers lie on different sides of the perpendicular bisector of b_1b_2.

Suppose we draw the perpendicular bisector for all $\binom{11}{2} = 55$ pairs of bells. Then, by the previous paragraph, the order in which a villager hears the bells is determined precisely by which region induced by these lines the villager lies in.

It is well-known that m lines in the plane create at most $\frac{m(m+1)}{2} + 1$ regions, and hence the perpendicular bisectors cannot produce more than $\frac{55\times 66}{2} + 1 = 55 \times 33 + 1 = 1816 < 2026$ regions. So there is some pair of villagers who share a region, i.e. there is some pair of villagers who hear the same order of tones. $\square$

Comment. The above argument does not give an optimal bound on the number of regions; one can get a better bound than $\frac{m(m+1)}{2} + 1$ by using the fact that perpendicular bisectors for a triple of points will meet at their circumcentre.

Problem 3. *(proposed by Andrija Živadinović)*

Let a and p be positive integers with p prime, and let $n = p^a$. Define $f : \mathbb{Z}^n \to \mathbb{Z}^n$ by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_1 + x_2 \\ x_1 + x_2 + x_3 \\ \vdots \\ x_1 + x_2 + x_3 + \cdots + x_n \end{pmatrix}.$$

Find, in terms of a and p, the smallest value of k such that, for every $x \in \mathbb{Z}^n$, n divides every component of $f^k(x) - x$. (Here, f^k denotes k iterated applications of f.)

Notes on Marking. None yet.

Solution 1. *(solution by Daniel Naylor)*

Let $\mathcal{M}_{n,n}(\mathbb{Z}/n\mathbb{Z})$ be the set of $n \times n$ matrices with all operations taken modulo n. Let $A \in \mathcal{M}_{n,n}(\mathbb{Z}/n\mathbb{Z})$ be the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Then the function f in the question is $x \mapsto Ax$, and the question asks us to find the least k such that $A^k = I$, where A^k is taken modulo n.

Let $B_m \in \mathcal{M}_{n,n}(\mathbb{Z}/n\mathbb{Z})$ be defined by

$$(B_m)_{ij} = \begin{cases} 1 & i - j = m \\ 0 & \text{otherwise} \end{cases}$$

Note that $B_m = 0$ for $m \geq n$ and that $B_m = B_1^m$ for $m \geq 0$. Also note that $I - B_1$ has determinant 1 (a unit in $\mathbb{Z}/n\mathbb{Z}$), so $I - B_1$ is invertible, so we have by the formula for a geometric progression that

$$A = B_1^0 + B_1^1 + \cdots + B_1^{n-1} = (I - B_1^n)(I - B)_1^{-1} = (I - B_1)^{-1}.$$

It now suffices to find the least value of k such that $(I - B_1)^k = I$.

Using the fact that $B_1^m = B_m$ and that $B_m = 0$ for $m \geq n$, we can calculate

$$(I - B_1)^k = \sum_{m=0}^{k} \binom{k}{m} (-1)^m B_1^m = I + \sum_{m=1}^{n-1} \binom{k}{m} (-1)^m B_m.$$

Since the B_m's involved in this sum are independent, we have that this sum equals I if and only if $\binom{k}{m}$ is a multiple of n for each $m = 1, \ldots, n-1$.

Lemma: A positive integer k is a multiple of $\frac{n^2}{p}$ if and only if it satisfies the property that $\binom{k}{m}$ is a multiple of n for each of $m = 1, \ldots, n-1$.

Proof. Suppose that k satisfies the second condition. Then by considering $m = 1$, we get that k is a multiple of n. Clearly the first condition also implies that k is a multiple of n, thus we can assume from now on that k is a multiple of n.

Note that if k is a multiple of n and $0 < m < n$, then for all $1 \le i \le m$ we have $\nu_p(k) > \nu_p(j)$ and hence $\nu_p(k-j) = \nu_p(j)$. Hence

$$\begin{aligned}
\nu_p\left(\binom{k}{m}\right) &= \nu_p(k) - \nu_p(m) \\
&\quad + \nu_p(k-1) - \nu_p(1) \\
&\quad + \nu_p(k-2) - \nu_p(2) \\
&\quad + \cdots \\
&\quad + \nu_p(k-m+1) - \nu_p(m-1) \\
&= \nu_p(k) - \nu_p(m) + 0 + 0 + \cdots + 0 \\
&= \nu_p(k) - \nu_p(m)
\end{aligned}$$

So we have that $\binom{k}{m}$ is a multiple of n if and only if $\nu_p(k) - \nu_p(m) \ge a$. In particular, the second condition in the Lemma holds if and only if $\nu_p(k) - \nu_p(m) \ge a$ for all $0 < m < n$, which is clearly equivalent to the first condition in the Lemma. □

Thus $(I - B_1)^k = I$ if and only if k is a multiple of $\frac{n^2}{p}$. So the answer to the question is $\frac{n^2}{p}$. □

Problem 4.

(proposed by Ittihad Hasib)

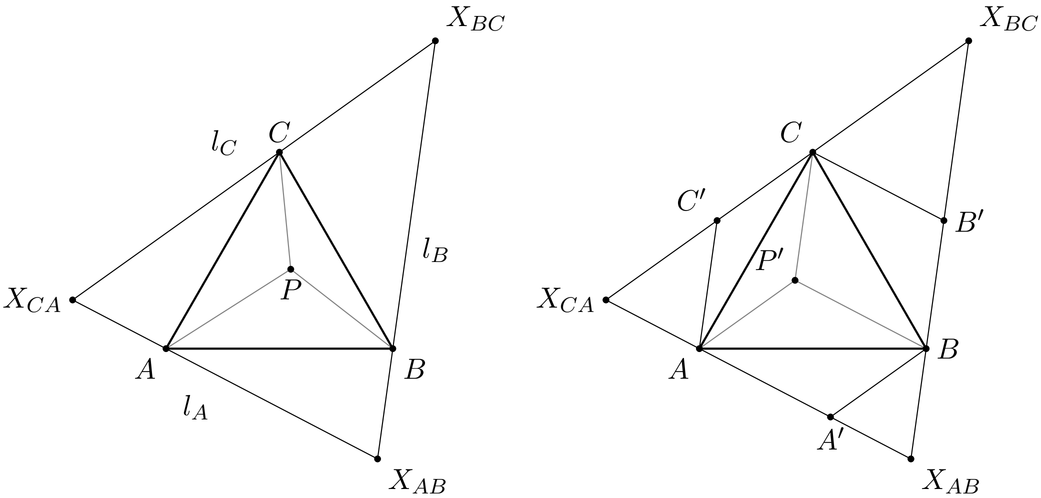
Let P be a point strictly inside an equilateral triangle ABC of area 1. Let ℓ_A be the result of rotating the line AP by 60° clockwise about A. Define ℓ_B and ℓ_C similarly. What is the minimal possible area of the triangle formed by ℓ_A, ℓ_B, and ℓ_C?

Notes on Marking. None yet.

Solution 1.

(solution by Daniel Naylor)

Let X_{AB} be the intersection of l_A with l_B, and define X_{BC}, X_{CA} similarly.



Let P' be the point that is obtained when we rotate P by 120° anticlockwise about the centre of ABC. Then the line l_A is the line passing through A that is parallel to PB, and similarly for l_B, l_C.

Let A' be the intersection of $X_{AB}X_{CA}$ with the line through B that is parallel to AP', and define B', C' similarly.

Clearly the hexagon $AA'BB'CC'$ has area 2 (by parallelograms). Also note that triangles $X_{BC}B'C$, $X_{CA}C'A$, $X_{AB}A'B$ are all similar to $X_{AB}X_{BC}X_{CA}$. Let the scale factors of similarity be $r_A, r_B, r_C < 1$ respectively. Then the area a of $X_{BC}X_{CA}X_{AB}$ satisfies the equation:

$$a = (r_1^2 + r_2^2 + r_3^2)a + 2,$$

thus a is minimised when $r_1^2 + r_2^2 + r_3^2$ minimised. Since $r_1 + r_2 + r_3 = 1$ (using the fact that $AA' = B'C$), we have $r_1^2 + r_2^2 + r_3^2 \geq \frac{1}{3}$, and hence $a \geq 3$. Equality is achieved when P is the centre of ABC. $\square$

Problem 5.

(proposed by Andrija Živadinović)

What is the largest positive integer n such that there exists a complex number $a \in \mathbb{C}$ for which the numbers $a, a^2, \ldots, a^n$ are all distinct and lie on a single non-degenerate parabola in the complex plane?

A parabola in the complex plane is a set of the form

$$\{x + yi \in \mathbb{C} : ax^2 + bxy + cy^2 + dx + ey + f = 0\},$$

where a*,* b*,* c*,* d*,* e*,* f*,* x*, and* y *are real numbers; and* a*,* b*, and* c *satisfy the equation* $b^2 - 4ac = 0$ *and are not all* 0*. The parabola is non-degenerate if and only if the polynomial* $ax^2 + bxy + cy^2 + dx + ey + f$ *does not factor into two polynomials of degree one. This description of a non-degenerate parabola corresponds to all rotations and dilations of the standard parabola* $y = x^2$ *in the Cartesian plane.*

Notes on Marking. None yet.

Comment. The following solution is quite technical, but the technicality is used for rigour. A similar solution exists without the language of algebraic geometry, although it is slightly longer. Fact 1 (or a similar fact specific to parabolas which is sufficient for this problem) can be derived by analysing the degrees of freedom of the general form of a parabola in the complex plane, and by substituting in five coordinates (x_i, y_i), and solving the resulting system of five linear equations.

Solution 1.

(solution by Andrija Živadinović and Daniel Naylor)

Let $\mathbb{P}^5 = (\mathbb{R}^6 \setminus \{0\})/\sim$, where $\sim$ is the equivalence relation defined by $x \sim y$ if and only if $x = \lambda y$ for some $\lambda \neq 0$. We equip $\mathbb{P}^5$ with the quotient topology inherited from $\mathbb{R}^6$.

Note that we can view elements of $\mathbb{P}^5$ as encoding a conic (by associating a conic with the equivalence of the tuple (a, b, c, d, e, f)). We will use this correspondence throughout the solution.

We begin by recalling / stating a pair of facts about conics (Facts 1 and 2), and then state without proof a straightforward topology exercise (Fact 3).

Fact 1. Given 5 distinct points on the plane with no four collinear, there exists a unique conic passing through them.

Moreover, if we let $S \subset \mathbb{C}^5$ be the set of tuples of 5 distinct points with no four collinear, and let $g : S \to \mathbb{P}^5$ be the function that sends a tuple of 5 points to the unique conic that passes through them, then g is continuous.

Fact 2. A conic is degenerate if and only if it contains a triple of collinear points (non-degenerate means that when the conic is written in the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$, the left hand side does not factor into two polynomials of degree one).

Fact 3. Given a continuous function $\gamma : [0, 1] \to \mathbb{P}^5$, there is a continuous function $\tilde{\gamma} : [0, 1] \to \mathbb{R}^6$ such that $\gamma = q_\sim \circ \tilde{\gamma}$, where $q_\sim$ is the quotient map $q_\sim : \mathbb{R}^6 \to \mathbb{P}^5$.

Claim. If $n \geq 6$, then there does not exist such an $a \in \mathbb{C}$.

Proof. Suppose P is a non-degenerate parabola passing through $a, a^2, \dots, a^6$. Note that P passes through $a^2, a^3, \dots, a^6$, but so does the distinct parabola $a \cdot P$ ($a \cdot P$ here means P rotated and scaled by a).

However, note that no three of $a^2, a^3, \dots, a^6$ are collinear, since the parabola P does not contain a triple of collinear points. Thus, by Fact 1, there is only one conic passing through them. This contradicts the previous paragraph. □

Claim. If $n = 5$, then there does exist such an $a \in \mathbb{C}$.

Proof. Let $s : \mathbb{C} \to \mathbb{C}^5$ be the function defined by $s(a) = (a, a^2, \dots, a^5)$, and let $\gamma : [0, 1] \to \mathbb{C}$ be defined by

$$\gamma(t) = (1 - t) \cdot 1.001i + t \cdot e^{0.5001i\pi}.$$

Note that this γ satisfies the following properties:

- $s \circ \gamma$ has image contained in S (so that the two bullet points below make sense);
- $g \circ s \circ \gamma(0)$ is a (degenerate) hyperbola (i.e. $b^2 - 4ac > 0$); and
- $g \circ s \circ \gamma(1)$ is an ellipse (i.e. $b^2 - 4ac < 0$).

Now we use this γ and its properties to show existence of a. Let $\tilde{\gamma}$ be the result of applying Fact 2 to $g \circ s \circ \gamma$, and define the function $h : [0, 1] \to \mathbb{R}$ by $h = k \circ \tilde{\gamma}$, where $k : \mathbb{R}^6 \to \mathbb{R}$ is defined by

$$k(a, b, c, d, e, f) = b^2 - 4ac.$$

Then $h(0) > 0$ and $h(1) < 0$, so by Intermediate Value Theorem there exists a $c \in (0, 1)$ such that $h(c) = 0$.

Note that $a = \gamma(c)$ certainly gives a parabola (since $h(c) = 0$), so it remains to check that the parabola is non-degenerate. But this follows by checking that no three of $a, a^2, \dots, a^5$ are collinear. □

Thus the largest possible value of n is 5. □