IMPERIAL-CAMBRIDGE
MATHEMATICS
COMPETITION

9th Edition (2025–2026)

ROUND TWO

Official Solutions*

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*A solution may receive full or partial marks even if it does not appear in this booklet.

Problem 1. *(proposed by Tony Wang)*

Prove that for all odd-degree polynomials p, there exist real numbers a and c and a non-zero real number d such that $f(a-d)+c = f(a) = f(a+d)-c$.

Notes on Marking. One mark was deducted for not analysing the case of f being linear in any solution where that was a special case.

For Solution 1, two marks were given for stating that any fixed non-zero real number d can be chosen. Furthermore, three marks were given for defining g, and one more for claiming g has an odd degree. Finally, one mark was given for calculating each of the coefficients of x^n and x^{n-1} in g, and two marks were given for calculating the coefficient of x^{n-2} in g.

Solution 1. *(solution by ICMC Committee)*

Note that the problem is trivial if f has degree 1. So assume from now on that the degree of f is greater than 2.

We claim that there is always a solution with $d = 1$. This is equivalent to saying that there exists a real number a such that $f(a-1) - f(a) = f(a+1) - f(a)$, since we can then set c to be this value.

Let $g(x) = f(x-1) + f(x+1) - 2f(x)$. By the above, it suffices to show that g has a root. To do this, we will show that g is an odd degree polynomial.

Suppose that $f(x)$ is of odd degree n ;and that

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \text{lower order terms}.$$

Then:

- The coefficient of x^n in $g(x)$ is $a_n + a_n - 2a_n = 0$.
- The coefficient of x^{n-1} in $g(x)$ is $(-na_n + a_{n-1}) + (na_n + a_{n-1}) - 2a_{n-1} = 0$.
- The coefficient of x^{n-2} in $g(x)$ is

  $$\left(\binom{n}{2}a_n - (n-1)a_{n-1} + a_{n-2}\right) + \left(\binom{n}{2}a_n + (n-1)a_{n-1} + a_{n-2}\right) - 2a_{n-2} = 2\binom{n}{2}a_n,$$

  which is non-zero.

Thus g is on odd-degree polynomial, as desired. □

Solution 2. *(solution by Contestants)*

We present an alternative proof that $g(x)$ from Solution 1 has a root. Without loss of generality we can assume f has a positive leading coefficient, as otherwise we can consider $-f$, choose the same values of a and d and $c' = -c$.

Since f'' is an odd-degree polynomial with positive leading coefficient, we can find $L, R \in \mathbb{R}$ such that $f''(x) < 0$ for $x < L$, and $f''(x) > 0$ for $x > R$. Then f is concave on the set $\{x \in \mathbb{R} : x < L\}$, so by definition, for $x < L - 1$, we have $g(x) < 0$. Similarly, we get $g(x) > 0$ for $x > R + 1$. As g is a polynomial, hence continuous, by the Intermediate Value Theorem g has a root, as required. □

Problem 2. *(proposed by Dylan Toh)*

On the Euclidean plane, there are 11 bells of distinct tones and 2026 stationary villagers. All 11 bells are simultaneously rung exactly once. Supposing that sound travels at a constant speed, and that no villager hears two or more bells at the same time, prove that there are two villagers who hear the same order of bell tones regardless of where the bells and the villagers are.

Notes on Marking. The most common deduction on this problem was due to arithmetic errors. Some contestants incorrectly recalled the formula for the maximum number of regions using n lines by omitting the $+1$, and 1 mark was deducted for this. When contestants used a solution based on Euler's formula, 1 mark was deducted if F was rewritten as $1 + E - V$ to be maximised, without reference to F anymore. Maximising $1 + E - V$ directly is difficult since $\max(E) = 2530$ and $\min(V) = 1$. Some contestants who used a double-counting argument lost marks for attributing the wrong number of edges or vertices to finite or infinite regions.

Solution 1. *(solution by ICMC Committee)*

A pair of villagers hear a different order if and only if they hear some pair in a different order. For any pair of bells, the order in which they are heard by a villager depends precisely on which side of the perpendicular bisector the villager is situated. Combining these two observations, we have that a pair of villagers hear a different order if and only if there is a pair of bells b_1, b_2 such that the villagers lie on different sides of the perpendicular bisector of b_1b_2.

Suppose we draw the perpendicular bisector for all $\binom{11}{2} = 55$ pairs of bells. Then, by the previous paragraph, the order in which a villager hears the bells is determined precisely by which region induced by these lines the villager lies in.

It is well-known that m lines in the plane create at most $\frac{m(m+1)}{2} + 1$ regions, and hence the perpendicular bisectors cannot produce more than $\frac{55\times 56}{2} + 1 = 55 \times 28 + 1 = 1541 < 2026$ regions. So there is some pair of villagers who share a region, i.e. there is some pair of villagers who hear the same order of tones. $\square$

Comment. The above argument does not give an optimal bound on the number of regions; one can get a better bound than $\frac{m(m+1)}{2} + 1$ by using the fact that perpendicular bisectors for a triple of points will meet at their circumcentre. Since there are $\binom{11}{3}$ triples of points, each of which has a unique circumcentre when the points are in general position, and each of which reduces the number of regions by 1, we can deduce that there are a maximum of 1376 regions.

Solution 2. *(solution by contestants)*

We follow Solution 1 except at the part where we need to count the number of regions formed by the 55 perpendicular bisectors. We will use Euler's formula – since $F - E + V = 2$, we have $F = 2 + E - V$.

Note that F is maximised when the bells and perpendicular bisectors are in general position. To prove this, place one villager in the interior of each existing region. Since there is some minimum distance between any villager and any perpendicular bisector, there is some $\epsilon > 0$ distance which we can move each bell without any perpendicular bisector touching any villager. Hence, by moving each bell and perpendicular bisector into general position, each villager will still have their own region, so we did not decrease the number of regions. Hence, we may assume without loss of generality that the bells and perpendicular bisectors are in general position.

In order to use Euler's formula, we first have to deal with the infinite edges. To do this, we draw a circle around all the intersections of any two lines, take the 110 intersection points as new vertices, and connect the new vertices with 110 edges. The portion of each line outside the circle is removed. This process adds exactly one region: each infinite region now becomes a finite region inside the "circle", and there is now one new infinite region outside of our graph.

Let us now count E and V. To count E, note that each perpendicular bisector intersects every other perpendicular bisector. Given the perpendicular bisector ℓ of bells A and B, for any other pair of bells C and D, it will intersect ℓ at a unique point. For any other bell C, the perpendicular bisectors of AC and BC will coincide at the circumcentre of the triangle $\triangle ABC$. Hence, the total number of intersections on ℓ is $\binom{9}{2}+9=45$. Hence, each line will be split into 46 line segments. Meanwhile, the circle adds 110 line segments, so $E=46\times 55+110=2640$. Meanwhile, the number of vertices is the number of unordered pairs of unordered pairs of bells $((A,B),(C,D))$, which is $\frac{1}{2}\binom{11}{2}\binom{9}{2}=990$, plus the number of circumcentres of three bells, which is $\binom{11}{3}=165$, plus the 110 extra vertices created by the circle, for a total of 1265. Hence, we have

$$F+1 \;=\; 2+2640-1265 \;=\; 1377,$$

which implies that $F=1376$. □

Solution 3. *(solution by contestants)*

We follow Solution 1 except at the part where we need to count the number of regions formed by the 55 perpendicular bisectors. We will double count the number of line segments and infinite rays, first via the number of intersections on each line, and secondly via the number of finite and infinite regions. Without loss of generality, we may assume that the 55 bisectors are distinct. We count the number of parts that each bisector is partitioned into by the other line segments. Consider the perpendicular bisector ℓ of two bells A and B – for each further pair of bells C and D, the perpendicular bisector will intersect ℓ at most once. Furthermore, for any other bell C, the perpendicular bisectors of AC and BC will coincide on ℓ. Hence, there will be a maximum of $\binom{9}{2}+9=45$ intersections on ℓ, partitioning ℓ into 44 line segments, and 2 infinite rays.

We can now establish an upper bound on the number of regions as follows. First, assume that not all perpendicular bisectors are parallel. Hence, no infinite regions will be half-planes, so each infinite region will border exactly 2 infinite rays and any number of finite line segments. Meanwhile, the boundary of each finite region will be composed of at least three finite line segments. Of course, each line segment and ray has two sides, and can contribute to the boundaries of two different regions. Since the total number of infinite rays is $55\times 2=110$, and the maximum number of finite line segments is $55\times 44=2420$, we will form exactly $2\times 110\div 2=110$ infinite regions and at most $2\times 2420\div 3=1613\frac{1}{3}$ finite regions, for a total maximum of $1723\frac{1}{3}$ regions. □

Solution 4. *(based on a solution by Luka Sebbag)*

We follow Solution 1 except at the part where we need to count the number of regions formed by the 55 perpendicular bisectors. We will double count the number of region corners by region and by intersection between two or more perpendicular bisectors. We will work on the real projective plane, so we can assume that any two lines meet at a point and any three points have a circumcentre. Finally, without loss of generality, we may assume that the 55 bisectors are distinct.

Let n_i for $i\in\{2,\ldots,55\}$ be the number of i-line intersections. The number of region corners created by an i-line intersection is $2i$, so to find the maximum number of region corners, we would like to find the maximum possible value of

$$C := \sum_{i=2}^{55} 2in_i, \qquad \text{given that} \qquad P := \sum_{i=2}^{55} \binom{i}{2} n_i = \binom{55}{2} \quad \text{and} \quad T := \sum_{i=2}^{55} \binom{b}{3} n_i \geq \binom{11}{3},$$

where b is the largest integer such that $\binom{b}{2} \leq i$. Here C stands for "corner", P stands for "pair intersections", and T stands for "triangle circumcentres". The first condition corresponds to the fact that there are at most $\binom{55}{2}$ intersections between pairs of lines, but at an i-line intersection, $\binom{i}{2}$ of them coalesce into a single intersection. The second condition corresponds to the fact that the three perpendicular bisectors of a triple of bells (of which there are $\binom{11}{3} = 165$) will coincide at the circumcentre of those bells, but if $b > 3$ bells are concyclic, it will show up as an $\binom{b}{2}$-line intersection, with $\binom{b}{3}$ circumcentres coalescing at a single point. Of course, having an $\binom{b}{2}$-line intersection doesn't necessarily mean that b points are concyclic – hence the inequalities – but the converse is true, so any set of 55 perpendicular bisectors generated by 11 bells must satisfy the second condition.

We want to maximise C, so increasing a particular n_i by 1 increases C by $2i$ while increasing P by $\frac{i(i-1)}{2}$, meaning that we only buy $\frac{4}{i-1}$ worth of C for each unit of P that we spend from our budget of $\binom{55}{2}$. For the second condition, note that increasing n_i by 1 where $\binom{b}{2} < i$ is strictly worse than increasing n_i where $\binom{b}{2} = i$, so let $i = \binom{b}{2}$. Increasing an $n_{\binom{b}{2}}$ by 1 to increase T buys $\binom{b}{3} = \frac{b(b-1)(b-2)}{6}$ units of T at the cost of

$$\binom{i}{2} = \frac{i(i-1)}{2} = \frac{\frac{b(b-1)}{2}\left(\frac{b(b-1)}{2} - 1\right)}{2} = \frac{b(b-1)(b+1)(b-2)}{8}.$$

units of P, which works out to $\frac{4}{3(b+1)}$ units of T per unit of P.

Hence, the cheapest way to bring T above 165 also satisfies the goal of maximising C. In other words, increasing $n_{\binom{b}{2}}$ for $b > 3$ by 1 instead of increasing n_3 by 1 is simultaneously a more expensive way to increase T and a more expensive way to increase C. Hence, the optimal distribution of n_i's that satisfies both conditions and maximises C is the distribution $n_3 = 165$, $n_2 = 990$, and $n_i = 0$ for all other i. Note that this is strictly better than any other distribution since it completely uses up the P budget of $\binom{55}{2}$, achieving the $T = 165$ equality as cheaply as possible and allocating the rest of the budget to increase C as cheaply as possible.

Hence, the maximum number of region corners we have is $4 \times 990 + 6 \times 165 = 4950$. We can assume that no infinite region is a half-plane (since otherwise all perpendicular bisectors are parallel and we have at most 56 regions), and hence each line is intersected by at least one other line. This implies that each line consists of exactly two infinite rays, for a total of 110 infinite rays. Since each infinite region requires exactly two infinite rays, and each infinite ray contributes to exactly two infinite regions, we have exactly 110 infinite regions, each of which uses up at least 1 region corner. All other regions are finite, so they use up at least 3 region corners. The region corners at infinity can be discarded. Since each region corner is used at most once, the maximum number of regions we can have is

$$110 + \frac{4950 - 110}{3} = 1723\frac{1}{3}. \qquad \square$$

Problem 3. *(proposed by Andrija Živadinović)*

Let a and p be positive integers with p prime, and let $n = p^a$. Define $f : \mathbb{Z}^n \to \mathbb{Z}^n$ by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_1 + x_2 \\ x_1 + x_2 + x_3 \\ \vdots \\ x_1 + x_2 + x_3 + \cdots + x_n \end{pmatrix}.$$

Find, in terms of a and p, the smallest value of k such that, for every $x \in \mathbb{Z}^n$, n divides every component of $f^k(x) - x$. (Here, f^k denotes k iterated applications of f.)

Notes on Marking. All solutions to this problem essentially consisted of two main steps: rephrasing the problem in terms of binomial coefficients, and solving the problem from there. We allocated 4 marks to the first step and 5 marks to the second step. The last mark was awarded for a correct answer.

In the first step, there were a few common sources of mark deductions. In solution 1, simply stating the formula without proving it resulted in a 2-mark deduction. Stating that it could be proven using induction but not mentioning the key steps (such as the hockey-stick identity), resulted in a 1-mark deduction. Showing that $f^k(x)_2 = kx_1 + x_2$ or that $f^k(x)_3 = \frac{k(k+1)}{2}x_1 + kx_2 + x_3$ was not worth any marks.

In the second step, we awarded a mark for proving that $n \mid k$, but not for proving that $n \leq k$. Many contestants also neglected to prove that $k = p^{2a-1}$ worked, and in some cases marks were deducted depending on the difficulty of proving this given what they have established.

In some solutions, an off-by-one error in one direction or the other – usually in the variable r as used in Solution 1 – caused cascading errors of varying degrees. Penalties were applied accordingly. Even when this resulted in $k = p^{2a-1} - 1$, the mark for a correct answer was not awarded.

Solution 1. *(solution by Tony Wang)*

We will first prove by induction on k that the i-th component of $f^k(x)$ will be

$$f^k(x)_i = \sum_{j=1}^{i} \binom{k-1+i-j}{i-j} x_j.$$

- **Base Case:** This is trivial since $k = 1$ implies each binomial coefficient evaluates to 1.
- **Inductive Step:** Suppose that the inductive hypothesis is true for $f^k(x)$. Then for

$f^{k+1}(x)$, we get

$$\begin{aligned}
f^{k+1}(x)_i &= \sum_{r=1}^{i} f^k(x)_r \\
&= \sum_{r=1}^{i}\sum_{j=1}^{r} \binom{k-1+r-j}{i-j} x_j \\
&= \sum_{j=1}^{i} x_j \left(\sum_{r=j}^{i} \binom{k-1+r-j}{r-j} \right) \\
&= \sum_{j=1}^{i} x_j \binom{k-1+i-j+1}{i-j} \qquad \text{(by the hockey-stick identity)} \\
&= \sum_{j=1}^{i} \binom{(k+1)-1+i-j}{i-j} x_j.
\end{aligned}$$

Now, letting $x = \mathbf{e}_1 \in \mathbb{Z}^n$, we have

$$f^k(x) - x = \begin{pmatrix} 0 \\ \binom{k}{1} \\ \binom{k+1}{2} \\ \vdots \\ \binom{k+n-2}{n-1} \end{pmatrix}.$$

This means that we need to find that smallest k such that

$$n = p^a \;\Bigg|\; \binom{k-1+r}{r} = \frac{(k+r-1)(k+r-2)\cdots(k+1)(k)}{r!}$$

for all $r \in \{1, 2, \ldots, n-1\}$. It is easy to see that this is also sufficient. In the case of $r = 1$, we get $n \mid \binom{k}{1} = k$, so let k be a multiple of p^a henceforth. Now define

$$g(r) := \frac{k+r-1}{r} \qquad \text{so that} \qquad \binom{k-1+r}{r} = g(1)g(2)g(3)\cdots g(r).$$

We want to prove that the p-adic valuation of $g(1)\cdots g(r)$ is equal to that of $g(1)$ except when $p \mid r$, where the p-adic valuation dips momentarily. To prove this, note that the only way to have $\nu_p(g(r)) = -b$ for some positive integer b is to have $p \mid r$. Since we know from before that $p \mid k$, we can deduce that $p \nmid k + r - 1$, and so $\nu_p(r) = b$. However, since $0 < r < p^a$, we have $\nu_p(r) < a \le \nu_p(k)$, and so

$$\nu_p(g(r+1)) = \nu_p\left(\frac{k+r}{r+1}\right) = \nu_p(k+r) = \nu_p(r) = b.$$

Hence, $\nu_p(g(r)g(r+1)) = 0$ when $p \mid r$. On the other hand, when $p \nmid r$ and $p \nmid r+1$, it is clear that $\nu_p(g(r)) = 0$. Hence, we can conclude that

$$\nu_p\big(g(1)\cdots g(r)\big) = \begin{cases} \nu_p(g(1)), & \text{if } p \nmid r \\ \nu_p(g(1)) - \nu_p(r), & \text{if } p \mid r \end{cases}.$$

This means that

$$\nu_p\left(\binom{k-1+r}{r}\right) \ge a \iff \nu_p(g(1)) - \nu_p(r) \ge a \qquad \forall r \in \{1, \ldots, n-1\}$$

Since $\nu_p(r) = a - 1$ when $r = p^{a-1}$, this forces $\nu_p(g(1)) = \nu_p(k) \geq 2a - 1$ and $k \geq p^{2a-1}$. Since $\nu_p(r) \leq a - 1$, we can see that $k = p^{2a-1}$ works. $\square$

Solution 2.

(solution by Daniel Naylor)

Let $\mathcal{M}_{n,n}(\mathbb{Z}/n\mathbb{Z})$ be the set of $n \times n$ matrices with all operations taken modulo n. Let $A \in \mathcal{M}_{n,n}(\mathbb{Z}/n\mathbb{Z})$ be the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Then the function f in the question is $x \mapsto Ax$, and the question asks us to find the least k such that $A^k = I$, where A^k is taken modulo n.

Let $B_m \in \mathcal{M}_{n,n}(\mathbb{Z}/n\mathbb{Z})$ be defined by

$$(B_m)_{ij} = \begin{cases} 1 & i - j = m \\ 0 & \text{otherwise} \end{cases}$$

Note that $B_m = 0$ for $m \geq n$ and that $B_m = B_1^m$ for $m \geq 0$. Also note that $I - B_1$ has determinant 1 (a unit in $\mathbb{Z}/n\mathbb{Z}$), so $I - B_1$ is invertible, so we have by the formula for a geometric progression that

$$A = B_1^0 + B_1^1 + \cdots + B_1^{n-1} = (I - B_1^n)(I - B)_1^{-1} = (I - B_1)^{-1}.$$

It now suffices to find the least value of k such that $(I - B_1)^k = I$.

Using the fact that $B_1^m = B_m$ and that $B_m = 0$ for $m \geq n$, we can calculate

$$(I - B_1)^k = \sum_{m=0}^{k} \binom{k}{m} (-1)^m B_1^m = I + \sum_{m=1}^{n-1} \binom{k}{m} (-1)^m B_m.$$

Since the B_m's involved in this sum are independent, we have that this sum equals I if and only if $\binom{k}{m}$ is a multiple of n for each $m = 1, \ldots, n - 1$.

> **Lemma:** A positive integer k is a multiple of p^{2a-1} if and only if it satisfies the property that $\binom{k}{m}$ is a multiple of n for each of $m = 1, \ldots, n - 1$.
>
> *Proof.* Suppose that k satisfies the second condition. Then by considering $m = 1$, we get that k is a multiple of n. Clearly the first condition also implies that k is a multiple of n, thus we can assume from now on that k is a multiple of n.
>
> Note that if k is a multiple of n and $0 < m < n$, then for all $1 \leq i \leq m$ we have $\nu_p(k) > \nu_p(j)$ and hence $\nu_p(k - j) = \nu_p(j)$. Hence
>
> $$\begin{aligned} \nu_p\left(\binom{k}{m}\right) &= \nu_p(k) - \nu_p(m) + \sum_{i=1}^{m-1} \nu_p(k - i) - \nu_p(i) \\ &= \nu_p(k) - \nu_p(m) \end{aligned}$$
>
> So we have that $\binom{k}{m}$ is a multiple of n if and only if $\nu_p(k) - \nu_p(m) \geq a$. In particular, the second condition in the Lemma holds if and only if $\nu_p(k) - \nu_p(m) \geq a$ for all $0 < m < n$, which is clearly equivalent to the first condition in the Lemma. $\square$

Thus $(I - B_1)^k = I$ if and only if k is a multiple of p^{2a-1}. So the answer to the question is p^{2a-1}. $\square$

Comment. Another way of completing the first part of the problem is to write the vector x as a generating function $x(y)$ in the ring $\mathbb{Z}[y]/\langle y^n \rangle$, and note that the function f sends $x(y)$ to

$$x(y)(1+y+y^2+\cdots) \;=\; \frac{x(y)}{1-y}.$$

Hence, f^k sends $x(y)$ to

$$x(y)(1+y+y^2+\cdots)^k \qquad \text{or} \qquad x(y)\frac{1}{(1-y)^k}.$$

Expanding the former expression using stars and bars yields the binomial coefficients seen in Solution 1, while expanding the latter expression yields the binomial coefficients seen in Solution 2. Indeed, the two conditions

$$n \;\Bigg|\; \binom{k-1+i}{i} \quad \forall i \in \{1,\ldots,n-1\} \qquad \text{and} \qquad n \;\Bigg|\; \binom{k-1}{i} \quad \forall i \in \{1,\ldots,n-1\}$$

are equivalent, as can be seen using Pascal's triangle. In the second part of the problem, many contestants also used Legendre's formula or Kummer's theorem (both the "sum of digits in base p" and the "number of carries in base p" formulations).

Problem 4.

(proposed by Ittihad Hasib)

Let P be a point strictly inside an equilateral triangle ABC of area 1. Let ℓ_A be the result of rotating the line AP by 60° clockwise about A. Define ℓ_B and ℓ_C similarly. What is the minimal possible area of the triangle formed by ℓ_A, ℓ_B, and ℓ_C?

Notes on Marking. Essentially all contestant solutions to this problem matched up roughly to one of the three solutions below. For Solution 2, the main variations were that some solutions used the lengths form of Ceva's Theorem, and some explicitly used barycentric coordinates. For Solution 2, the most common variation was to use the sine rule for area calculations.

Many contestants attempted to use Lagrange multipliers. Marks were only awarded if the calculation was complete or if it involved progress resembling one of the solutions below.

The marks for this problem were split in the following way:

(a) 1 mark for stating that the minimal possible area is 3.

(b) 6 marks for setting up some constrained variables and an expression for the area in terms of these variables. To earn any marks for this part, the resulting minimisation had to be a significant simplification of the original problem. The expression and constraint had to be fully correct in order to earn all 6 marks.

(c) 3 marks for minimising the expression from (b).

Unfortunately, quite a few contestants made calculation mistakes in (b), particularly for attempts following a "ratio of areas" approach as in Solution 2. This was usually given a 2-4 mark deduction since it meant that the script was automatically unable to address (c). The second most common type of mistake was to give an incorrect proof of the Claim in Solution 3.

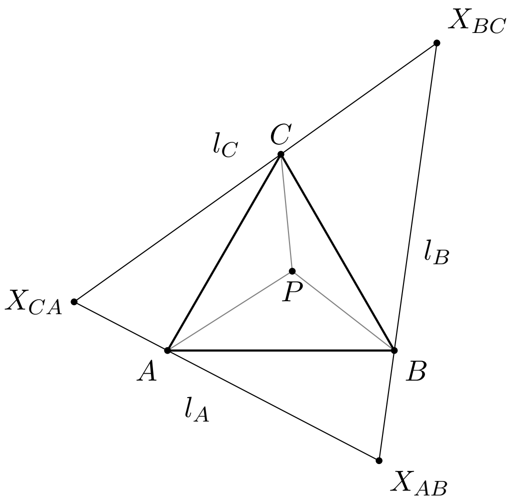
In cases where the expression and constraint in (b) was particularly easy to minimise, for example as in Solution 1, 7 marks were awarded instead of 6.

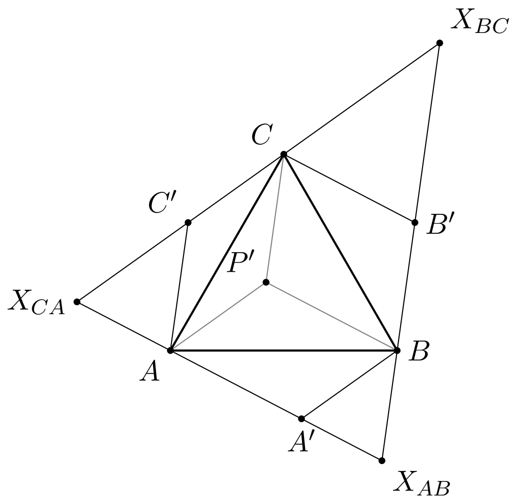
Many contestants obtained an expression for the area in terms of angles and constrained it using trigonometric Ceva; such attempts did not receive the marks for (b), since the resulting problem is about as hard as the original problem (and in fact, no contestant managed to make this approach work, despite the fact that quite a few observed it).

Solution 1.

(solution by Daniel Naylor)

Let X_{AB} be the intersection of l_A with l_B, and define X_{BC}, X_{CA} similarly.





Let P' be the point that is obtained when we rotate P by $120°$ anticlockwise about the centre of ABC. Then the line l_A is the line passing through A that is parallel to PB, and similarly for l_B, l_C.

Let A' be the intersection of $X_{AB}X_{CA}$ with the line through B that is parallel to AP', and define B', C' similarly.

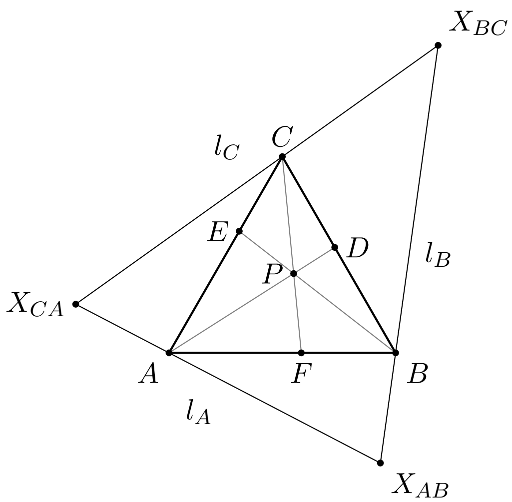
Clearly the hexagon $AA'BB'CC'$ has area 2 (by parallelograms). Also note that triangles $X_{BC}B'C$, $X_{CA}C'A$, $X_{AB}A'B$ are all similar to $X_{AB}X_{BC}X_{CA}$. Let the scale factors of similarity be $r_A, r_B, r_C < 1$ respectively. Then the area a of $X_{BC}X_{CA}X_{AB}$ satisfies the equation:

$$a = (r_1^2 + r_2^2 + r_3^2)a + 2,$$

thus a is minimised when $r_1^2 + r_2^2 + r_3^2$ minimised. Since $r_1 + r_2 + r_3 = 1$ (using the fact that $AA' = B'C$), we have $r_1^2 + r_2^2 + r_3^2 \geq \frac{1}{3}$, and hence $a \geq 3$. Equality is achieved when P is the centre of ABC. □

Solution 2. *(solution by contestants)*

Let X_{AB} be the intersection of l_A with l_B, and define X_{BC}, X_{CA} similarly. Let D be the intersection of AP with BC, let E the intersection of BP with CA and let F be the intersection of CP with AB.



Note that APE is similar to $AX_{AB}B$ because

$$\angle PAE = 60° - \angle BAP = \angle X_{AB}AB$$

and

$$\angle AEP = \angle AEB = 120° - \angle EBA = \angle ABX_{AB}.$$

Given any triangle XYZ, let $[XYZ]$ denote its area. The above similarity allows us to calculate:

$$\begin{aligned}[AX_{AB}B] &= \frac{|AB|^2}{|AE|^2}[APE] \\ &= \frac{|AB|^2}{|AE|^2}\frac{|AE|}{|AC|}[APC] \\ &= \frac{|AB|}{|AE|}[APC]\end{aligned}$$

Define $x = [BCP]$, $y = [CAP]$ and $z = [ABP]$. Note that $x+y+z = 1$ since $[ABC] = 1$. Since $\frac{|EC|}{|AE|} = \frac{[CEB]}{[EAB]} = \frac{[CEP]}{[EAP]}$, we have that

$$\frac{|EC|}{|AE|} = \frac{[CEB] - [CEP]}{[EAB] - [EAP]} = \frac{x}{z}.$$

Now we continue the earlier calculation:

$$[AX_{AB}B] = \frac{|AB|}{|AE|}[APC] = \left(1 + \frac{|EC|}{|AE|}\right) y = y + \frac{xy}{z}.$$

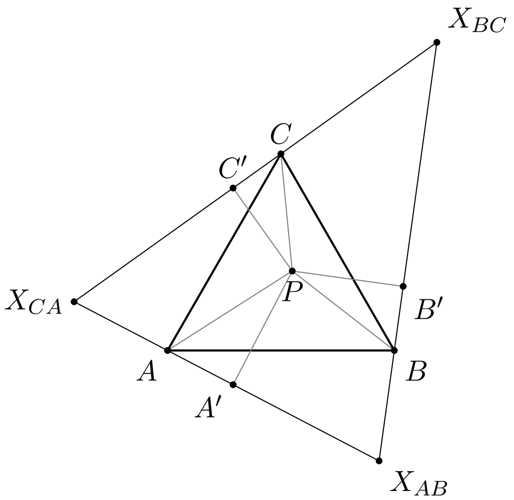
Similarly, we get $[BX_{BC}C] = z + \frac{yz}{x}$ and $[CX_{CA}A] = x + \frac{zx}{y}$. Then by AM-GM,

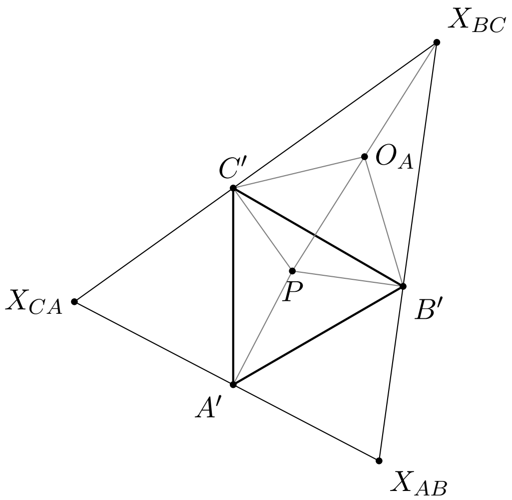
$$\begin{aligned}
[X_{BC}X_{CA}X_{AB}] &= [ABC] + [AX_{AB}B] + [BX_{BC}C] + [CX_{CA}A] \\
&= 1 + (x+y+z) + \frac{1}{2}\left(\frac{xy}{z} + \frac{xz}{y}\right) + \frac{1}{2}\left(\frac{yz}{x} + \frac{yx}{z}\right) + \frac{1}{2}\left(\frac{zx}{y} + \frac{zy}{x}\right) \\
&\geq 2 + x + y + z \\
&= 3
\end{aligned}$$

Equality holds when P is the centre of ABC, so 3 is the minimal possible area. □

Solution 3. *(solution by contestants)*

Let X_{AB} be the intersection of l_A with l_B, and define X_{BC}, X_{CA} similarly. Consider a rotation by 30° anticlockwise about P, followed by a dilation of scale factor $\frac{\sqrt{3}}{2}$. Let A', B' and C' be the images of A, B and C respectively.





Note that A' lies on $X_{AB}X_{CA}$ because PAA' is congruent to half an equilateral triangle, and hence $\angle A'AP = 60°$. Similarly for B', C'. Note also that PA' is perpendicular to $X_{CA}X_{AB}$.

Let $d_{A'B'}(X)$ denote the distance from X to $A'B'$, but with negative sign if X is on the same side of $A'B'$ as C'. Define $d_{B'C'}, d_{C'A'}$ similarly, and define $p_A = -d_{B'C'}(P)$, $p_B = -d_{C'A'}(P)$ and $p_C = -d_{A'B'}(P)$. Note that

$$\frac{1}{2}(p_A + p_B + p_C)|A'B'| = [A'B'C'] = \left(\frac{\sqrt{3}}{2}\right)^2 [ABC] = \frac{3}{4}$$

so $p_A + p_B + p_C = \frac{3}{2|A'B'|}$ (throughout this solution, $[XYZ]$ is the area of triangle XYZ).

Let O_A be the midpoint of PX_{BC}. O_A is in fact the circumcentre of $PB'X_{BC}C'$ because of the $90°$ angles. Note that

$$d_{B'C'}(X_{BC}) = 2d_{B'C'}(O_A) - d_{B'C'}(P) = 2d_{B'C'}(OA) + p_A.$$

Let $\alpha = \angle B'PC'$, $\beta = \angle C'PA'$ and $\gamma = \angle A'PB'$. By using the fact that angle at the circumcentre equals twice the angle at the circumference, the definition of $d_{B'C'}$ and splitting into cases according to whether or not $\alpha - 90°$ is positive, we get

$$d_{B'C'}(O_A) = \frac{1}{2}|B'C'|\tan(\alpha - 90°).$$

Using the fact that X_{BC} is on the opposite side of $B'C'$ to A', we can calculate $[B'X_{BC}C]$:

$$\begin{aligned}[B'X_{BC}C] &= \frac{1}{2}|B'C'|d_{B'C'}(X_{BC}) \\ &= \frac{1}{2}|B'C'|(2d_{B'C'}(OA) + p_A) \\ &= \frac{1}{2}|B'C'|(|B'C'|\tan(\alpha - 90°) + p_A) \\ &= \frac{1}{2}|B'C'|^2\tan(\alpha - 90°) + |B'C'|p_A\end{aligned}$$

Note that $\frac{3}{4} = [A'B'C'] = \frac{\sqrt{3}}{4}|A'B'|^2$, so $|A'B'|^2 = \sqrt{3}$. Hence

$$\begin{aligned}[X_{BC}X_{CA}X_{AB}] &= [A'B'C'] + \frac{1}{2}|A'B'|^2(\tan(\alpha - 90°) + \tan(\beta - 90°) + \tan(\gamma + 90°)) \\ &\quad + \frac{1}{2}|A'B'|(p_A + p_B + p_C) \\ &= \frac{3}{4} + \frac{1}{2}|A'B'|^2(\tan(\alpha - 90°) + \tan(\beta - 90°) + \tan(\gamma + 90°)) + \frac{3}{4} \\ &= \frac{3}{2} + \frac{\sqrt{3}}{2}(\tan(\alpha - 90°) + \tan(\beta - 90°) + \tan(\gamma + 90°))\end{aligned}$$

Let $x = \alpha - 90°$, $y = \beta - 90°$ and $z = \gamma - 90°$. Note that $60° < \alpha < 180°$, hence $-30° < x < 90°$, and similarly for y and z. Also note that $\alpha + \beta + \gamma = 360°$, so $x + y + z = 90°$.

Claim. For any $-30° \leq x, y, z < 90°$ satisfying $x + y + z = 90°$, we have

$$\tan x + \tan y + \tan z \geq 3\tan 30° = \sqrt{3}.$$

Proof. Note that the inequality holds if one of x, y or z exceeds $89.999°$. Therefore we may restrict to (x, y, z) satisfying $-30° \leq x, y, z \leq 89.999°$ and $x + y + z = 90°$, which gives a compact set. Since the left hand side is a continuous function of (x, y, z) on a compact set, it has a minimum. Let (x, y, z) be a witness for this minimum. we must show that the inequality holds for this triple.

If $(x, y, z) = (30°, 30°, 30°)$, then the inequality holds. Now suppose $(x, y, z) \neq (30°, 30°, 30°)$. We will show that (x, y, z) does not achieve the minimum on the set.

Without loss of generality, suppose $x = \max(x, y, z)$ and $z = \min(x, y, z)$. The constraints then give that $30° < x < 90°$ and $|z| \leq 30°$, so $\tan' x > \tan' z$ (where $\tan'$ means the derivative of tan). Then for any $\varepsilon > 0$ small enough, we have that $(x - \varepsilon, y, z + \varepsilon)$ also satisfies the constraints, but yields a smaller value, hence (x, y, z) does not achieve the minimum on the set. □

Finally, we can use the claim and earlier calculations to get that

$$[X_{BC}X_{CA}X_{AB}] \geq \frac{3}{2} + \frac{3}{2} = 3.$$

Equality is achieved when P is the centre of ABC, since in that case we have $x = y = z = 30°$. □

Problem 5. *(proposed by Andrija Živadinović)*

What is the largest positive integer n such that there exists a complex number $a \in \mathbb{C}$ for which the numbers $a, a^2, \ldots, a^n$ are all distinct and lie on a single non-degenerate parabola in the complex plane?

A parabola in the complex plane is a set of the form

$$\{x + yi \in \mathbb{C} : ax^2 + bxy + cy^2 + dx + ey + f = 0\},$$

where a, b, c, d, e, f, x, and y are real numbers; and a, b, and c satisfy the equation $b^2 - 4ac = 0$ *and are not all* 0. *The parabola is non-degenerate if and only if the polynomial* $ax^2 + bxy + cy^2 + dx + ey + f$ *does not factor into two polynomials of degree one. This description of a non-degenerate parabola corresponds to all rotations and dilations of the standard parabola* $y = x^2$ *in the Cartesian plane.*

Notes on Marking. Marks were split into two main steps in the proof of this problem. No marks were awarded for stating the correct answer.

The first 4 marks could be obtained for proving, explicitly or implicitly, that the solution is bounded above by $n \leq 5$. Partial marks for attempts at proving this specific upper bound could be earned. No marks were awarded for any attempts proving cases of $n < 5$ were possible, in other words for proving that there exists a lower bound.

The second 6 marks were awarded for showing that the bound could be reached.

The two pathways were either one of using the intermediate value theorem rigorously or of constructing a valid a satisfying the condition to $n = 5$, corresponding to the solutions below, respectively.

For the proof using the IVT, a total of 3 marks could be awarded for finding two points drawing out a parabola and an ellipse, respectively. 2 marks were awarded for parametrising a curve linking these two points and stating that the discriminant changes continuously on the curve, so as to use the IVT. 1 mark was awarded for showing that no three points drawn will be collinear at any such intermediate value on the curve. If no points drawing either a hyperbola or an ellipse were identified, no subsequent points were awarded, even for correct subsequent deductions.

For the constructive proof 1 mark was awarded for writing down the correct system of equations to solve for the parabola parameters, 2 marks could be earned for solving said system, 1 mark could be earned for using this solution to simplify the discriminant condition to one unknown parametrising a, and finally solving this condition to find the final value of a. Partial marks were awarded for any attempt, stated or not to be an attempt at a constructive proof, that aligned with the steps of this proof path.

Comment. The following solution is quite technical, but the technicality is used for rigour. A similar solution exists without the language of algebraic geometry. Indeed, one can construct a solution using just the proof to claim 1 in Solution 1, and Solution 2. Fact 1 (or a similar fact specific to parabolas which is sufficient for this problem) can be derived by analysing the degrees of freedom of the general form of a parabola in the complex plane, and by substituting in five coordinates (x_i, y_i), and solving the resulting system of five linear equations, similarly to how it is done in Solution 2.

Solution 1. *(solution by Andrija Živadinović and Daniel Naylor)*

Let $\mathbb{P}^5 = (\mathbb{R}^6 \setminus \{0\})/\sim$, where $\sim$ is the equivalence relation defined by $x \sim y$ if and only if $x = \lambda y$ for some $\lambda \neq 0$. We equip $\mathbb{P}^5$ with the quotient topology inherited from $\mathbb{R}^6$.

Note that we can view elements of $\mathbb{P}^5$ as encoding a conic (by associating a conic with the equivalence of the tuple (a, b, c, d, e, f)). We will use this correspondence throughout the solution.

We begin by recalling / stating a pair of facts about conics (Facts 1 and 2), and then state without proof a straightforward topology exercise (Fact 3).

Fact 1. Given 5 distinct points on the plane with no four collinear, there exists a unique conic passing through them.

Moreover, if we let $S \subset \mathbb{C}^5$ be the set of tuples of 5 distinct points with no four collinear, and let $g : S \to \mathbb{P}^5$ be the function that sends a tuple of 5 points to the unique conic that passes through them, then g is continuous.

Fact 2. A conic is degenerate if and only if it contains a triple of collinear points (non-degenerate means that when the conic is written in the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$, the left hand side does not factor into two polynomials of degree one).

Fact 3. Given a continuous function $\gamma : [0, 1] \to \mathbb{P}^5$, there is a continuous function $\tilde{\gamma} : [0, 1] \to \mathbb{R}^6$ such that $\gamma = q_\sim \circ \tilde{\gamma}$, where $q_\sim$ is the quotient map $q_\sim : \mathbb{R}^6 \to \mathbb{P}^5$.

Claim. If $n \geq 6$, then there does not exist such an $a \in \mathbb{C}$.

Proof. Suppose P is a non-degenerate parabola passing through $a, a^2, \ldots, a^6$. Note that P passes through $a^2, a^3, \ldots, a^6$, but so does the distinct parabola $a \cdot P$ ($a \cdot P$ here means P rotated and scaled by a).

However, note that no three of $a^2, a^3, \ldots, a^6$ are collinear, since the parabola P does not contain a triple of collinear points. Thus, by Fact 1, there is only one conic passing through them. This contradicts the previous paragraph. □

Claim. If $n = 5$, then there does exist such an $a \in \mathbb{C}$.

Proof. Let $s : \mathbb{C} \to \mathbb{C}^5$ be the function defined by $s(a) = (a, a^2, \ldots, a^5)$, and let $\gamma : [0, 1] \to \mathbb{C}$ be defined by

$$\gamma(t) = (1 - t) \cdot 1.001i + t \cdot e^{0.5001i\pi}.$$

Note that this γ satisfies the following properties:

- $s \circ \gamma$ has image contained in S (so that the two bullet points below make sense);
- $g \circ s \circ \gamma(0)$ is a (degenerate) hyperbola (i.e. $b^2 - 4ac > 0$); and
- $g \circ s \circ \gamma(1)$ is an ellipse (i.e. $b^2 - 4ac < 0$).

Now we use this γ and its properties to show existence of a. Let $\tilde{\gamma}$ be the result of applying Fact 2 to $g \circ s \circ \gamma$, and define the function $h : [0, 1] \to \mathbb{R}$ by $h = k \circ \tilde{\gamma}$, where $k : \mathbb{R}^6 \to \mathbb{R}$ is defined by

$$k(a, b, c, d, e, f) = b^2 - 4ac.$$

Then $h(0) > 0$ and $h(1) < 0$, so by Intermediate Value Theorem there exists a $c \in (0, 1)$ such that $h(c) = 0$.

Note that $a = \gamma(c)$ certainly gives a parabola (since $h(c) = 0$), so it remains to check that the parabola is non-degenerate. But this follows by checking that no three of $a, a^2, \ldots, a^5$ are collinear. □

Thus the largest possible value of n is 5. $\square$

Solution 2.

(solution by Gergely Rozgonyi)

This solution aims only to provide a constructive proof for $n = 5$ once the upper bound of $n \leq 5$ is established by some method, akin or not to the above solution.

Claim. $a = \dfrac{3+\sqrt{5}}{2} e^{i\pi/3}$ will provide a valid five-term power sequence on a parabola.

There are multiple reasons to motivate this ansatz. First, notice that that $e^{i\pi/2}$ would lead to three collinear points in the first five powers of a, so it is not valid for a parabola (or conic), and we must use a smaller distinguished angle, in this case 60° to be taken for a. Taking a distinguished angle in general then ensures that the exact coordinates of the points can be written down.

Secondly, without knowing the radius *a priori*, the system to solve for the parameters of the parabola will have a "nice" structure if $a \propto e^{i\pi/3}$, as the middle of the power sequence, a^3, is purely real, and the other two pairs of points will be reflections of each other across the origin up to a scaling factor dependent on the modulus of a.

Now let $a = re^{i\pi/3}$, $r \neq 0$. The resulting powers of a are as follows:

$$\begin{gathered} a = \frac{1}{2}r + i\frac{\sqrt{3}}{2}r, \qquad a^2 = -\frac{1}{2}r^2 + i\frac{\sqrt{3}}{2}r^2, \qquad a^3 = -r^3, \\ a^4 = -\frac{1}{2}r^4 - i\frac{\sqrt{3}}{2}r^4, \qquad a^5 = \frac{1}{2}r^5 - i\frac{\sqrt{3}}{2}r^5. \end{gathered} \tag{1}$$

Since we will have five equations for six unknowns, we can pick $F = -r^8$ without loss of generality (as a parabola is invariant to uniform scaling of its six parameters), noting that this is not necessary and was done simply to make the following algebra easier to follow. The resulting system of equations is as follows:

$$\begin{pmatrix} \frac{1}{4}r^2 & \frac{\sqrt{3}}{4}r^2 & \frac{3}{4}r^2 & \frac{1}{2}r & \frac{\sqrt{3}}{2}r \\ \frac{1}{4}r^4 & -\frac{\sqrt{3}}{4}r^4 & \frac{3}{4}r^4 & -\frac{1}{2}r^2 & \frac{\sqrt{3}}{2}r^2 \\ r^6 & 0 & 0 & -r^3 & 0 \\ \frac{1}{4}r^8 & \frac{\sqrt{3}}{4}r^8 & \frac{3}{4}r^8 & -\frac{1}{2}r^4 & -\frac{\sqrt{3}}{2}r^4 \\ \frac{1}{4}r^{10} & -\frac{\sqrt{3}}{4}r^{10} & \frac{3}{4}r^{10} & \frac{1}{2}r^5 & -\frac{\sqrt{3}}{2}r^5 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \end{pmatrix} = \begin{pmatrix} r^8 \\ r^8 \\ r^8 \\ r^8 \\ r^8 \end{pmatrix}. \tag{2}$$

The above system can be greatly simplified thanks to the nice structure alluded to earlier by defining the following auxiliary variables:

$$\begin{aligned} \alpha &= \frac{1}{4}\left(A + B\sqrt{3} + 3C\right), \qquad & \gamma &= \frac{1}{2}\left(D + E\sqrt{3}\right), \\ \beta &= \frac{1}{4}\left(A - B\sqrt{3} + 3C\right), \qquad & \delta &= \frac{1}{2}\left(D - E\sqrt{3}\right), \end{aligned} \tag{3}$$

which lead to the following two systems of two equations *each*:

$$\begin{cases} \alpha r^2 + \gamma r = r^8 \\ \alpha r^8 - \gamma r^4 = r^8 \end{cases} \qquad \begin{cases} \beta r^4 - \delta r^2 = r^8 \\ \beta r^{10} + \delta r^5 = r^8 \end{cases} \tag{4}$$

with the third line of the original system

$$Ar^6 - Dr^3 = r^8 \tag{5}$$

acting as a "coupling" required to solve for all five parameters of the parabola from the auxiliary equations (3). Equation (4) can be readily solved to yield

$$\alpha = r^3, \qquad \beta = r, \qquad \gamma = r^7 - r^4, \qquad \delta = r^3 - r^6, \tag{6}$$

which can be substituted back into equations (3) and (5) to give the following values for the parameters A, B, C, D and E:

$$\begin{aligned} A &= 1 - r + r^2 - r^3 + r^4, \\ B &= \frac{2}{\sqrt{3}} r(r^2 - 1), \\ C &= -\frac{1}{3}(1 - 3r + r^2 - 3r^3 + r^4), \\ D &= r^3 - r^4 - r^6 + r^7, \\ E &= -\frac{1}{\sqrt{3}}(r^3 + r^4 - r^6 - r^7). \end{aligned} \tag{7}$$

Before proceeding to use the consistency condition $B^2 - 4AC = 0$ to solve for r, we make the following observation: B^2 will be palindromic centred on r^4 and $4AC$ is a product of palindromic polynomials each centred on r^2, so we expect the consistency condition to yield a palindromic polynomial in r centred on r^4 as well. As it will become apparent below, this is a key property associated with the ansatz $a \propto e^{i\pi/3}$ that will allow finding the roots of the 8^{th} degree consistency condition and hence the required r.

Expanding the consistency condition it can be found that

$$\frac{3}{4}(B^2 - 4AC) = 1 - 4r + 6r^2 - 8r^3 + 7r^4 - 8r^5 + 6r^6 - 4r^7 + r^8 \equiv C(r). \tag{8}$$

Since it is as we expected palindromic, by the usual procedure and some more algebra we can rewrite it as

$$\frac{C(r)}{r^4} = u^4 - 4u^3 + 2u^2 + 4u - 3 = (u-1)^2(u+1)(u-3), \qquad u = r + \frac{1}{r}, \tag{9}$$

where the final equality can be found using standard factorisation methods. Converting back from u to r we find that the factorisation of the 8^{th} degree consistency polynomial is

$$C(r) = (r^2 - r + 1)^2(r^2 + r + 1)(r^2 - 3r + 1) = 0, \tag{10}$$

the positive real roots of which are $r = \dfrac{3 \pm \sqrt{5}}{2}$ proving the claim. $\square$