2018/19 Imperial Mathematics Competition

Individual Competition
Round Two (Final)

Sunday, 10 March 2019
Time allowed: 3 hours

Please fill in the lines below, writing your name in BLOCK CAPITALS.

Name: ______________________________

Candidate ID: ______________________________

Institution: ______________________________

Signature: ______________________________

Instructions:

- DO NOT open the paper until told to do so by the invigilator.
- This paper consists of 4 questions. Each question carries 10 marks.
- You may use black or blue pens only. Write clearly.
- Use separate answer sheets for each question. On each answer sheet, write down the question number, your initials and candidate ID. Use both sides wherever possible.
- You must prove any assertions you make that are not reasonably expected to be taught in or before the first year of an undergraduate degree programme in mathematics.
- Cross out any rough work that you would not like to be marked.
- The invigilators will collect your answer sheets as well as this cover sheet and the question paper.

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1. Observe that, in the usual chessboard colouring of the two-dimensional grid, each square has 4 of its 8 neighbours black and 4 white. Does there exist a way to colour the three-dimensional grid such that each cube has half of its 26 neighbours black and half white? Is this possible in four dimensions?

2. In the symmetric group S_n ($n \geq 3$), let $G_{a,b}$ be the subgroup generated by the 2-cycle $(a\ b)$ and the n-cycle $(1\ 2\ \cdots\ n)$. Find the index $|S_n : G_{a,b}|$.

3. Show that if the faces of a tetrahedron have the same area, then they are congruent.

4. Let $f : \{0,1\}^n \to \{0,1\} \subseteq \mathbb{R}$ be a function. Call such a function a Boolean function. Let $\wedge$ denote the component-wise multiplication in $\{0,1\}^n$. For example, for $n = 4$, $(0,0,1,1) \wedge (0,1,0,1) = (0,0,0,1)$.

Let $S = \{i_1, i_2, \cdots, i_k\} \subseteq \{1, 2, \cdots, n\}$. f is called the oligarchy function over S if

$$f(x) = x_{i_1} x_{i_2} \cdots x_{i_k} \text{ (with the usual multiplication)},$$

where x_i denotes the i-th component of x. By convention, f is called the oligarchy function over $\varnothing$ if f is constantly 1.

(i) Suppose f is not constantly zero. Show that f is an oligarchy function if and only if f satisfies

$$f(x \wedge y) = f(x)f(y), \quad \forall x, y \in \{0,1\}^n.$$

Let Y be a uniformly distributed random variable over $\{0,1\}^n$. Let T be an operator that maps Boolean functions to functions $\{0,1\}^n \to \mathbb{R}$, such that

$$(Tf)(x) = \mathrm{E}_Y(f(x \wedge Y)), \quad \forall x \in \{0,1\}^n$$

where $\mathrm{E}_Y()$ denotes the expectation over Y. f is called an eigenfunction of T if $\exists\, \lambda \in \mathbb{R}\backslash\{0\}$ such that

$$(Tf)(x) = \lambda f(x), \quad \forall x \in \{0,1\}^n.$$

(ii) Prove that f is an eigenfunction of T if and only if f is an oligarchy function.

END OF QUESTIONS

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for creating or suggesting the questions for the Individual Competition.

Another one of the questions appeared in a 1994 article by Alexander Shen, titled "Entrance Examinations to the Mekh-mat". The question was attributed to Nesterenko (1974); it was allegedly used in the USSR against "undesirable" students. The article was published in *Mathematical Intelligencer*, 16 (4), 6-10.

We would like to thank our friends at Imperial College London for proofreading and trying out the questions, among whom Thomas Groves has gone further by contributing a question. We are sorry that we have not been able to use other questions created by members of the Department of Mathematics and would like to thank them for their understanding and support.

Imperial Mathematics Competition