IMPERIAL COLLEGE
MATHEMATICS
COMPETITION

ROUND ONE

Sunday, 20 December 2020

Instructions:

- As far as possible, the exam should be taken under test conditions, in a quiet room, with only the permitted materials and a printout of the exam if desired.
- You will have 3 hours to solve 6 problems, each of which carries 10 marks. You will have 30 minutes after the competition to upload your solutions.
- You are recommended to use a black or blue pen or a dark pencil. Rulers, compasses, and erasers may be used but will not be required. All electronic devices, including calculators, are prohibited, except to view this paper, email our committee, or type up your solutions in a word processor or TeX editor.
- Write your Contestant ID on every page. We will require a separate file upload for each problem. Do not include any other personally identifiable information such as your name in your scripts or filenames.
- Problems are listed roughly in order of difficulty. Proofs are expected for all problems even if they only ask for an answer.

If you encounter difficulties or wish to ask a question about the paper or the problems, please contact us at `contest@icmathscomp.org`. We will try to get back to you as quickly as possible.

Problem 1. A set of points in the plane is called *sane* if no three points are collinear and the angle between any three distinct points is a rational number of degrees.

(a) Does there exist a countably infinite sane set $\mathcal{P}$?

(b) Does there exist an uncountably infinite sane set $\mathcal{Q}$?

Problem 2. Let A be a square matrix with entries in the field $\mathbb{Z}/p\mathbb{Z}$ such that $A^n - I$ is invertible for every positive integer n. Prove that there exists a positive integer m such that $A^m = 0$.

(A matrix having entries in the field $\mathbb{Z}/p\mathbb{Z}$ means that two matrices are considered the same if each pair of corresponding entries differ by a multiple of p.)

Problem 3. Let $s_n = \displaystyle\int_0^1 \sin^n(nx)\,dx$.

(a) Prove that $s_n \leq \dfrac{2}{n}$ for all odd n.

(b) Find all the limit points of the sequence $s_1, s_2, s_3, \ldots$.

Problem 4. Does there exist a set $\mathcal{R}$ of positive rational numbers such that every positive rational number is the sum of the elements of a unique finite subset of $\mathcal{R}$?

Problem 5. Find all composite positive integers m such that, whenever the product of two positive integers a and b is m, their sum is a power of 2.

Problem 6. There are $n+1$ squares in a row, labelled from 0 to n. Tony starts with k stones on square 0. On each move, he may choose a stone and advance the stone up to m squares where m is the number of stones on the same square (including itself) or behind it.

Tony's goal is to get all stones to square n. Show that Tony cannot achieve his goal in fewer than $\frac{n}{1} + \frac{n}{2} + \cdots + \frac{n}{k}$ moves.