IMPERIAL COLLEGE
MATHEMATICS
COMPETITION

2021–2022

ROUND ONE

Official Solutions *

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*A solution may receive full or partial marks even if it does not appear in this booklet.

Problem 1.

(proposed by Constantinos Papachristoforou)

Let T_n be the number of non-congruent triangles with positive area and integer side lengths summing to n. Prove that $T_{2022} = T_{2019}$.

Notes on Marking. There were two solutions in general: counting solutions were more popular than bijection solutions. For bijection solutions, 4 points were awarded for stating the bijection and 6 points for proving it. Partial bijection solutions either did not consider the triangle inequality or triangles with side length 1. Both of these mistakes would have cost the candidate 3 marks each, however stating it without proof would result in losing 2 marks instead. Candidates attempting a counting solution would struggle to not overcount if they did not first order the side lengths. A common incorrect approach was using a map with the shortest side fixed to deal with triangles having a side length of 1. No marks were given for these maps as they are not well-defined.

Solution 1.

(solution by Constantinos Papachristoforou)

Let $a \leq b \leq c$ be positive integer sides of a triangle with perimeter 2019. It must have $c < a+b$ and $a \geq 1$. There is a corresponding triangle with sides $(a+1), (b+1), (c+1)$ and perimeter 2022. This is a triangle as $(c+1) < (a+1)+(b+1)$ and $(a+1) \geq 1$. Therefore, $T_{2019} \leq T_{2022}$.

Similarly, let $x \leq y \leq z$ be positive integer sides of a triangle with perimeter 2022. It must have $z < x+y$ and $x \geq 1$. There is a corresponding triangle with sides $(x-1), (y-1), (z-1)$ and perimeter 2019. Prove it is a triangle by contradiction:

1. Suppose that $(z-1) \geq (x-1)+(y-1)$. Since $z < x+y$, this implies that $z = x+y-1$. Adding z to both sides gives $2z = 2021$. Contradiction. Therefore, $(z-1) < (x-1)+(y-1)$.

2. Suppose that $(x-1) < 1$. Since $x \geq 1$, this implies that $x = 1$. Substituting this into $(z-1) < (x-1)+(y-1)$ gives $z < y$. Contradiction. Therefore, $(x-1) \geq 1$.

Hence, $T_{2022} \leq T_{2019}$ and, therefore, $T_{2022} = T_{2019}$. □

Solution 2.

(solution by Simeon Kiflie)

Let $a \leq b \leq c$ be positive integer sides of a triangle with perimeter 2019. It must have $673 \leq c \leq 1009$ and $\left\lceil \frac{2019-c}{2} \right\rceil \leq b \leq c$ as

$$a \leq b \leq c \implies a+b \leq 2c \implies 2019 = a+b+c \leq 3c \implies 673 \leq c,$$

$$c < a+b \implies 2c < a+b+c = 2019 \implies c < \frac{2019}{2} \implies c \leq 1009,$$

$$a \leq b \implies 2019-c = a+b \leq 2b \implies \frac{2019-c}{2} \leq b \implies \left\lceil \frac{2019-c}{2} \right\rceil \leq b.$$

T_{2019} can be computed directly by counting each possible pair of b and c. Therefore,

$$T_{2019} = \sum_{c=673}^{1009} \left(c - \left\lceil \frac{2019-c}{2} \right\rceil + 1 \right) = \sum_{c=673}^{1009} \left(\left\lfloor \frac{3(c-673)}{2} \right\rfloor + 1 \right).$$

Similarly, let $x \le y \le z$ be positive integer sides of a triangle with perimeter 2022. It must have $674 \le z \le 1010$ and $\lceil \frac{2022-z}{2} \rceil \le y \le z$. Therefore,

$$T_{2022} = \sum_{z=674}^{1010} \left(\left\lfloor \frac{3(z-674)}{2} \right\rfloor + 1 \right) \implies T_{2022} = T_{2019} = \sum_{n=0}^{336} \left(\left\lfloor \frac{3n}{2} \right\rfloor + 1 \right) = 85177.$$

□

Solution 3.

(solution by contestants)

Let S_n be the number of positive integer triples $x \le y \le z$ summing to n. Ravi substitution shows that $a \le b \le c$ are the sides of a triangle if and only if there exists a unique triple of positive reals $x \le y \le z$ such that $a = x + y$, $b = x + z$ and $c = y + z$. If $a + b + c = 2022$, then $2(x+y+z) = 2022 \implies x+y+z = 1011$. If a, b, c are positive integers, then x, y, z are positive integers since $x = 1011 - c$, $y = 1011 - b$, $z = 1011 - a$ and $c < 1011$. Therefore, $T_{2022} = S_{1011}$. Similarly, if $a+b+c = 2019$, then $(x + \frac{1}{2}) + (y + \frac{1}{2}) + (z + \frac{1}{2}) = 1011$. Also, if a, b, c are positive integers, then $(x+\frac{1}{2}), (y+\frac{1}{2}), (z+\frac{1}{2})$ are positive integers. Therefore, $T_{2019} = S_{1011} = T_{2022}$. □

Problem 2. *(proposed by Tony Wang)*

Find all integers n for which there exists a table with n rows, 2022 columns, and integer entries, such that subtracting any two rows entry-wise leaves every remainder modulo 2022.

Notes on Marking. A correct answer was worth 1 mark. We condoned omitting integers less than 2 as answers. A construction for the $n = 2$ case was not awarded any marks, however the omission of a construction from an otherwise complete solution was penalised 1 mark. There are several "WLOG" arguments to be made, however we only awarded marks (specifically, 1 mark) to the argument that any constant could be added to all the entries in a column without changing the relevant properties of the table. We did not award marks for proving any other upper bounds on n.

Solution 1. *(solution by Tony Wang)*

The answer is $n = 1, 2$. We can construct a table with 2 rows as follows:

$$\begin{array}{cccccc} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 2 & 3 & \cdots & 2021. \end{array}$$

Now for the sake of contradiction suppose that $n \geq 3$. Let the sum of the i-th row be S_i. Note that $0 + 1 + \ldots + 2021 \equiv 1011 \pmod{2021}$, so we must have $S_i - S_j \equiv 1011 \pmod{2022}$ for all $i \neq j$. But then

$$0 \equiv (S_1 - S_2) + (S_2 - S_3) + (S_3 - S_1) \equiv 3033 \pmod{2022},$$

a contradiction. $\square$

Solution 2. *(solution by contestants)*

We prove that the answer is $n = 1, 2$, and use the construction as in the above solution. Now suppose for the sake of contradiction that $n \geq 3$. Since we can add a constant to each column without changing the relevant properties, we can assume WLOG that the first row is all zeroes. Then the second and third rows must contain 1011 even numbers and 1011 odd numbers, and the difference between the second and third rows must also have 1011 even numbers and 1011 odd numbers. However, supposing there are n positions where the second and third row are both even, then there must be $1011 - n$ positions where the second row is even and the third row is odd, and vice versa, and finally there must be n positions where the second and third row are both odd. Hence, there are $2n$ positions where the difference between the second and third rows is even, a contradiction.

Comment. Perhaps surprisingly, it is an open problem to find the maximum number of rows when 2022 is replaced by 25. More generally, it is conjectured that the maximum number of rows with c columns is equal to the smallest prime divisor of c. To the problem proposer's knowledge, this has been proven only when c is even, prime, or equal to 9 (by computer search).

Problem 3. *(proposed by Ethan Tan)*

Let $\mathcal{M}$ be the set of $n \times n$ matrices with integer entries. Find all $A \in \mathcal{M}$ such that $\det(A+B)+\det(B)$ is even for all $B \in \mathcal{M}$.

Notes on Marking. Many candidates guessed the right answer and worked and/or over F_2 earning 1 point. For solution 1, considering B with the same row as A was worth 3 points. For solution 3, noticing that the determinant of A is 0 is worth 2 points and starting an induction was awarded 1 more. Many candidates lost points for failing to justify some arguments. Some candidates attempted to use Jordan Normal form which did not work. Hence few marks were awarded for such attempts.

Solution 1. *(solution by Julian Yu)*

All solutions begin by turning the matrices into matrices from F_2. After turning into F_2, the problem now asks us to find all matrices in F_2 such that $\det(A+B) = \det(B)$ for all $B \in M_n(F_2)$. We shall prove that the only matrix A satisfying this condition is $A = O_n$.

Assume that A has a non-zero row. Consider the matrix B having the same non-zero row as A. We complete B by extending this to a basis of vectors containing the non-zero row.[†] Thus, the matrix B will have $\det(B) = 1$, while $\det(A+B) = 0$ due to $A+B$ having a zero row. Thus, this implies that $A = O_n$ in F_2, so A must have all entries even in $M_n(\mathbb{Z})$. $\square$

Solution 2. *(solution by Cristi Calin)*

As before, we have the problem written in F_2. Since triangularization doesn't necessarily work in this field, we apply the cyclic decomposition theorem instead. Let $A = C(f_1) \bigoplus C(f_2) \bigoplus \cdots C(f_k)$, where $f_k = X^k + a_{k-1}X^{k-1} + ... + a_0 I$ and

$$C(f_k) = \begin{pmatrix} 0 & 0 & ... & 0 & -a_0 \\ 1 & 0 & ... & 0 & -a_1 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 1 & -a_{k-1} \end{pmatrix}.$$

The problem is now solved through block matrices. For each $C(f_k)$, we take an X_k of the

$$X(f_k) = \begin{pmatrix} 1 & 0 & ... & 0 \\ 1 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & ... & 1 & 1 \end{pmatrix}.$$

The determinant of $C(f_k)+X(f_k)$ must be 1, since $\det(X(f_k)) = 1$, and $\det(X) = \prod_k \det(X(f_k))$. Thus, $1 - a_{k-1} = 1$, so $a_{k-1} = 0$.

We proceed inductively. Assume that, up until a_j, all $a-s$ are 0 in $C(f_k)$. We therefore take the matrix $X(f_k) = I + C$, where $C[i+1][i] = 1$ up until $i+1 = j$, and afterwards only 0. $C(f_k) + X(f_k)$ will be equal to $\left(\begin{smallmatrix} I & S \\ O & D \end{smallmatrix}\right)$, where D will have 1 on and under the main diagonal and 0 in rest, except for the upper-right term, which will be $-a_j$. The determinant of this matrix will be equal to $1 + a_j$ by Laplace expansion, and it will have to be equal to 1, so

[†]Or, suppose that the non-zero row is i, and entry j in that row is 1. Then we use the B which is an identity matrix when the i-th row and j-th column are removed.

$a_j = 0$, and the induction is complete. Thus, $f_k = X^k$, and the matrix A is nilpotent. From here, taking $X(f_k)$ to have 1 on and immediately above the main diagonal, we will get that $\det(X(f_k) + C(f_k)) = \det(X(f_k)) = 1 = 0$, which implies that $f_k = 0$. Thus, $A = O_n$, and A has only even entries in $M_n(\mathbb{Z})$. $\square$

Solution 3.

(solution by Alex Bosinta)

As before, we can work over $\mathbb{F}_2$ since nothing changes in the parity of a determinant if we add an even number to any element of the matrix. We will show that in fact we must have $A = O_n$, by induction on the size of the minors of A. Firstly, by setting $B = O_n$, we observe that $\det(A)$ has to be 0, so the unique n-sized minor of A must be 0.

Now assume by induction that we proved that all the minors of size $n, n-1, \dots, k$ of A are 0 for some $k \in \{2, 3, \dots, n\}$. Now let us choose a minor of A of size $k-1$ represented by the coefficients of A after removing the rows $i_1, i_2, \dots, i_{n-k+1}$ and columns $j_1, j_2, \dots, j_{n-k+1}$. Choose B to be the matrix with a 1 on positions (i_s, j_s) for all $s \in \{1, 2, \dots, n-k+1\}$ and 0 in all other positions. Then since $k \geq 2$ we must have that $n-k+1 \leq n-1$, so $\det(B)$ is 0. Denote the elements of B to be $b_{i,j}$. When we calculate $\det(A+B)$ (and keeping in mind that we work over $\mathbb{F}_2$) we get

$$\begin{aligned}
&\sum_{\sigma \in S_n} \prod_{i=1}^{n} (a_{i,\sigma(i)} + b_{i,\sigma(i)}) \\
&= \sum_{S \subset \{1,\dots,n-k+1\}} \sum_{\substack{\sigma \in S_n \\ \sigma(i_s)=j_s \forall s \in S \\ \sigma(i_s) \neq j_s \forall s \in \{1,\dots,n-k+1\} \setminus S}} \prod_{i=1, i \in S}^{n} (a_{i,\sigma(i)} + 1) \prod_{i=1, i \notin S}^{n} a_{i,\sigma(i)} \\
&= \sum_{S \subset \{1,\dots,n-k+1\}} \sum_{\substack{\sigma \in S_n \\ \sigma(i_s)=j_s \forall s \in S \\ \sigma(i_s) \neq j_s \forall s \in \{1,\dots,n-k+1\} \setminus S}} \sum_{T \subset S} \prod_{i=1, i \notin T}^{n} a_{i,\sigma(i)} \\
&= \sum_{T \subset \{1,\dots,n-k+1\}} \sum_{T \subset S \subset \{1,\dots,n-k+1\}} \sum_{\substack{\sigma \in S_n \\ \sigma(i_s)=j_s \forall s \in S \\ \sigma(i_s) \neq j_s \forall s \in \{1,\dots,n-k+1\} \setminus S}} \prod_{i=1, i \notin T}^{n} a_{i,\sigma(i)} \\
&= \sum_{T \subset \{1,\dots,n-k+1\}} \sum_{\substack{\sigma \in S_n \\ \sigma(i_s)=j_s \forall t \in T}} \prod_{i=1, i \notin T}^{n} a_{i,\sigma(i)}.
\end{aligned}$$

For each $T \subset \{1, \dots, n-k+1\}$, the term of the sum above is actually the determinant of the minor when we exclude lines i_s and columns j_s for all $s \in T$. This would be a minor of size $n - |T|$, hence we can assume it is 0 for any T of size smaller than or equal to $n-k$ by induction. This leaves us with only the term for $T = \{1, \dots, n-k+1\}$, which gives us the minor of size $k-1$ that we were calculating. On the other hand, the problem tells us that $\det(A+B) + \det(B)$ is 0 and since $\det(B) = 0$, we must have that this minor of size $k-1$ is also 0. Thus all minors of size $k-1$ are 0, so the induction is complete. Hence this proves that all minors of size 1 are 0, which are precisely all the elements of A, so $A = O_n$ in $\mathbb{F}_2$. Hence the solutions are all the matrices with only even elements. $\square$

Problem 4.

(proposed by Harun Khan)

Let p be a prime number. Find all subsets $S \subseteq \mathbb{Z}/p\mathbb{Z}$ such that

- if $a, b \in S$, then $ab \in S$, and
- there exists an $r \in S$ such that for all $a \in S$, we have $r - a \in S \cup \{0\}$.

Note: $\mathbb{Z}/p\mathbb{Z}$ denotes the integers modulo p.

Notes on Marking. One point was awarded for a **complete** set of solutions. However, in many cases 1 point was lost due to missing at least one solution. The fact that $r - a$ could be 0 (and not necessarily in S) was very important and cost many candidates 2 to 4 points. Some candidates lost 2 points if they forgot that r could in fact be 0. While many students correctly realized that if 0 is not in S and S works, then $S \cup \{0\}$ also works, but this does not work in reverse ($\{0, 1, -1\}$ always works for odd primes, but $\{1, -1\}$ never works). A few students did not realize that the multiplicative group of $\mathbb{Z}/p\mathbb{Z}$ was cyclic, or that S had to also be cyclic, something which made any proof messier, but still doable. One point was awarded for this observation. Some candidates earned 2 points for finding all solutions when $r = 0$.

Answer. $S = \{0\}, \{1\}, \{0, 1\}, (\mathbb{Z}/p\mathbb{Z})^\times$, and any S such that $0, -1 \in S$ and $S \setminus \{0\}$ is cyclic in $(\mathbb{Z}/p\mathbb{Z})^\times$. Note that we take $r = 0$ for all the solutions except $\{1\}$, $\{0, 1\}$, and $(\mathbb{Z}/p\mathbb{Z})^\times$ for which we take $r = 1$ instead.

Solution 1.

(solution by Harun Khan)

Let T be the set of working S given before. Notice $S \setminus \{0\}$ must be cyclic as it is closed under multiplication. If $0 \notin S$, then $S \cup \{0\}$ also works. So assume $0 \in S$ (so that $x \mapsto r - x$ fixes S).

Suppose $S \setminus \{0\}$ is non-empty (otherwise either S is empty, which trivially doesn't work, or $S = \{0\}$, which trivially works). Let $S = \{a_1, a_2, \ldots, a_m\}$ and $\Sigma = a_1 + a_2 + \cdots + a_m$. Since $S = r - S$, we have $\Sigma \equiv mr - \Sigma \pmod{p}$, so $mr \equiv 2\Sigma \pmod{p}$. We now have two cases:

- **Case 1:** $\Sigma \equiv 0 \pmod{p}$. Then either $m \equiv 0 \pmod{p}$ or $r \equiv 0 \pmod{p}$. If $m \equiv 0 \pmod{p}$, then S is either $\emptyset$ or $\mathbb{Z}/p\mathbb{Z}$. Note that $S = (\mathbb{Z}/p\mathbb{Z})^\times$ also works. Otherwise $r \equiv 0 \pmod{p}$. If $S = \{0\}$, we are done. Otherwise $S \setminus \{0\}$ is a subgroup of $(\mathbb{Z}/p\mathbb{Z})^\times$ containing -1, so $S = \{0, 1, g, g^2, \ldots\}$ where the order of g is even.
- **Case 2:** $\Sigma \not\equiv 0 \pmod{p}$. Then for $a \in S \setminus \{0\}$, $aS = S$, so $a\Sigma = \Sigma$, which implies $a = 1$. Hence S is either $\{1\}$ or $\{0, 1\}$. □

Solution 2.

(solution by Alex Bosinta & Ethan Tan)

The first condition forces $S \setminus \{0\}$ to be cyclic in $(\mathbb{Z}/p\mathbb{Z})^\times$. If $r = 0$, then since $1 \in S \setminus \{0\}$ we get $-1 \in S$. Otherwise, assume r is nonzero. If $x \in S \setminus \{0, 1\}$, then

$$\begin{aligned} x \in S \implies x^{-1} &\in S \\ \implies rx^{-1} &\in S \\ \implies r - rx^{-1} &\in S \\ \implies rx - r &\in S \\ \implies x - 1 &\in S. \end{aligned}$$

Hence $S \setminus \{0\}$ takes the form $\{1, \ldots, n\}$ for some $n \in \{1, \ldots, p-1\}$. $n = 1$ is already covered; otherwise, $\sum_{x \in S \setminus \{0\}} x = 0$ (by sum of roots of unity), so p divides $n(n+1)/2$, which forces $p \mid n$ (impossible) or $p \mid n+1$. Hence $n = p-1$, so $S = (\mathbb{Z}/p\mathbb{Z})^\times$ or $S = \mathbb{Z}/p\mathbb{Z}$. $\square$

Solution 3.

(solution by contestants)

We take a similar approach to the second solution. If $S \setminus \{0\}$ is empty, then either $S = \emptyset$ (which doesn't work) or $S = \{0\}$ (which does work). Otherwise, we note that $S \setminus \{0\}$ is a cyclic subgroup of $(\mathbb{Z}/p\mathbb{Z})^\times$. If $r = 0$, then since $1 \in S\{0\}$, we get $-1 \in S$, and all such subsets work (cyclic which contain 0 and -1).

If r is nonzero we shall prove first that $-1 \in S$. If $r = -1$ we are done. Now assume $r \neq -1$. First note that for any $a \in S \setminus \{0, r^{-1}\}$ we have $a^{-1} \in S \setminus \{0, r\}$, hence

$$r - a^{-1} \in S, \qquad \text{thus} \qquad a(r - a^{-1}) = ar - 1 \in S$$

If $r = 1$ then the above becomes $a - 1 \in S$ for any $a \in S \setminus \{0, 1\}$, while condition two from the statement gives us that $1 - a \in S$. If $S \setminus \{0, 1\}$ is empty, then we get the new solutions $S = \{1\}$ and $S = \{0, 1\}$, which both work. Also $a \neq 1$, so $a - 1 \neq 0$, hence

$$-1 = (a-1)^{-1}(1-a) \in S.$$

If $r \neq 1$, then we have $r^2 - 1 \in S$ and $r - 1 \in S$ by setting $a = r$ and $a = 1$ respectively above. Then we get that

$$r + 1 = (r-1)^{-1}(r^2 - 1) \in S,$$

thus $-1 = r - (r+1) \in S \cup \{0\}$.

Since $-1 \in S$, we have $-r \in S$. We shall prove by induction that $-ir \in S$ for any $i \in \{1, 2, \ldots, p-1\}$. The case $i = 1$ is already done. Now assume for some $i \in \{1, 2, \ldots, p-2\}$ that $-ir \in S$. Then $r + ir = r - (-ir) \in S \cup \{0\}$. $r + ir = 0$ if and only if $i = p - 1$, which is false by assumption. Hence $(i+1)r \in S$ and since $-1 \in S$, we get that $-(i+1)r \in S$. Induction is complete.

Now we know that for $r \neq 0$ we have $\{-r, -2r, \ldots, -(p-1)r\} \subseteq S$, but since these are all different elements, they must be all nonzero classes mod p, hence $S = \mathbb{Z}/p\mathbb{Z}$ or $S = (\mathbb{Z}/p\mathbb{Z})^\times$. $\square$

Problem 5.
(proposed by Ethan Tan)

A *tanned vector* is a nonzero vector in $\mathbb{R}^3$ with integer entries. Prove that any tanned vector of length at most 2021 is perpendicular to a tanned vector of length at most 100.

Notes on Marking. Many candidates attempted to construct the components of the vector using $(b, -a, 0)$ or the subspace spanned by $(b, -a, 0)$ and $(0, c, -b)$ or similar where (a, b, c) is the original tanned vector. There were no complete solutions with this approach or approaches using Bezout's lemma nor does there seem to be one and hence 0 was awarded in many cases. Some candidates earned 1 point by noticing that the difference of two vectors in a sphere of radius 50 lies in a sphere of radius 100. A common partial was to bound $|v \cdot u|$ with Cauchy Schwarz and earned candidates 3 points.

Solution 1.
(solution by Ethan Tan)

Let S_n be the set of nonzero tanned vectors of length at most n. Suppose for a contradiction that $\exists v \in S_{2021}$ that is not perpendicular to any vector in S_{100}. Consider $u \neq w \in S_{50}$. Then $u - w \in S_{100}$. By our assumption, $v \cdot (u - w) \neq 0 \implies v \cdot u \neq v \cdot w$. Then $v \cdot u$ takes distinct integer values for $u \in S_{50}$; so $\exists u \in S_{50}$ s.t. $|v \cdot u| \geq \frac{1}{2}|S_{50}|$. Hence

$$2021 \geq |v| \geq \frac{|v \cdot u|}{|u|} \geq \frac{1}{50}|v \cdot u| \geq \frac{1}{100}|S_{50}|. \tag{†}$$

Place cubes of side 1, one centred at every element of $S_{50} \cup \{\mathbf{0}\}$, to get a solid T with volume $|S_{50}| + 1$. For any u, the unit cube with centre u must contain a lattice point w with $|w| \leq |u| + d(u, v) \leq |u| + \frac{1}{2}\sqrt{3}$, so T contains the sphere with centre $\mathbf{0}$ and radius $50 - \frac{1}{2}\sqrt{3} > 49$. So

$$|S_{50}| + 1 = \mathrm{Vol}(T) \geq \frac{4}{3}\pi(49)^3 > 4 \cdot 49^3 > 2021 \times 100 + 1,$$

contradicting (†). $\square$

Comment. Solution 1 shows that in fact one can replace 100 with 74. By a more sophisticated argument one can reduce this further (noting that dot products of vertices in a sphere by a given vector cannot actually take the form $\{-n, 1 - n, \ldots, n\}$ since a sphere is not evenly distributed in any direction; using cross-sections can give better bounds on the largest dot product).

Solution 2.
(solution by contestants)

As in Solution 1, we assume that there exists a $v \in S_{2021}$ that is not perpendicular to any vector in S_{100}, and show that this implies

$$|S_{50}| \leq 2021 \cdot 100 = 202100. \tag{†}$$

Let T be the set of tanned vectors (x, y, z) such that $|x| \leq 29, |y| \leq 29$, and $|z| \leq 29$, except for $|x| = |y| = |z| = 29$. It is easy to see that the longest vector in T has length $\sqrt{29^2 + 29^2 + 28^2} < 50$, so $T \subset S_{50}$. Therefore $|S_{50}| \geq |T| = 59^3 - 9 > 202101$, contradicting (†). $\square$

Problem 6. *(proposed by Ethan Tan)*

Is it possible to cover a circle of area 1 with finitely many equilateral triangles whose areas sum to 1.01, all pointing in the same direction?

Notes on Marking. No marks were awarded for stating the correct answer, however, candidates stating that the answer was no were usually awarded 0 points. Thinking about Sierpinski coverings (e.g. by drawing a Sierpinski triangle) was worth 1 point and was a fairly common partial. Reducing the problem to tiling an upside-down triangle was worth 2 points. Solution 1 was more common than solution 2. For solution 2, 2 points were awarded by attempting to pack triangles inside the circle so that the remaining area goes to zero.

Solution 1. *(solution by Ethan Tan)*

We show that this is possible. Tile the plane with equilateral triangles of side δ, and take the set of triangles that touch the circle. This gives a cover of triangles (not necessarily pointing in the same direction) whose area approaches 1 as $\delta \to 0$. So it suffices to show that for all $\varepsilon > 0$ we can cover a down-pointing equilateral T with up-pointing triangles of total area $(1+\varepsilon)\text{Area}(T)$.

If T can be covered with $(1+x)\text{Area}(T)$, then split T into four equal-area equilateral triangles (the middle triangle points up and the other three point down). By placing the middle triangle and then covering the three down-pointing triangles we see that T can be covered in

$$(1 + 3(1 + x))\frac{\text{Area}(T)}{4} = \left(1 + \frac{3x}{4}\right)\text{Area}(T).$$

Starting with any cover and iterating this process gives the desired result. $\square$

Solution 2. *(solution by Ethan Tan)*

In this solution we show that the problem remains true when we replace equilateral triangles and a circle with any regions P, Q (assuming they are non-null measurable and bounded).

There exists a square of nonzero area contained inside P; suppose that the area of P is $\alpha \geq 1$ times the area of this square. P can be contained inside a large square; say P has area $\beta \leq 1$ times that of this square. For any $\varepsilon > 0$, we can find a finite, disjoint set of squares contained strictly inside Q whose area is $(1-\varepsilon)$ times that of Q, by measurability. Place a copy of P inside each of these squares, so that the total area covered is $(1-\varepsilon)\beta \times \text{Area}(Q)$. So we can remove these copies from Q to get Q_1, whose area is δ times that of Q (where $\delta = 1 - (1-\varepsilon)\beta < 1$).

We can apply the same process to Q_1 to get Q_2, and so on. The area of Q_n decreases at the rate of a geometric progression and so converges to 0. Finally, Q_n can be covered with a set of squares whose area is $(1+\varepsilon)\text{Area}(Q_n)$, and so can be covered with copies of P with total area $\alpha(1+\varepsilon)\text{Area}(Q_n)$.

Hence, it is possible to cover Q with copies of P of total area

$$\text{Area}(Q) - \text{Area}(Q_n) + \alpha(1+\varepsilon)\text{Area}(Q_n) = \text{Area}(Q) + [\alpha(1+\varepsilon) - 1]\text{Area}(Q_n).$$

Since $\text{Area}(Q_n) \to 0$ as $n \to \infty$, the RHS goes to $\text{Area}(Q)$, as required. $\square$