IMPERIAL COLLEGE

MATHEMATICS

COMPETITION

2021–2022

ROUND TWO

Official Solutions*

Last updated: 30 April 2022, 10pm

*A solution may receive full or partial marks even if it does not appear in this booklet.

Problem 1. *(proposed by Tony Wang)*

Let S be a set of 2022 lines in the plane, no two parallel, no three concurrent. S divides the plane into finite regions and infinite regions. Is it possible for all the finite regions to have integer area?

Notes on Marking. No marks were given for a correct answer by itself. Candidates lost 2 points for an incorrect proof that the area of a polygon with rational coordinates is rational.

Solution 1. *(solution by Ming Yean Lim)*

We show that the answer is yes. Arbitrarily choose 2022 lines L_i passing through two rational points such that no two are parallel and no three concurrent. Then any two such pair of lines intersect at a point with rational coordinates. The finite regions are polygons whose vertices have rational coordinates, hence they have rational area.* Now we may clear denominators by rescaling.

Concretely, let d be the lowest common multiple of the denominators of the areas and suppose that the line L_i is given by $a_i x + b_i y = c_i$. Then define L'_i by $(a_i/d)x + (b_i/d)y = c_i$. Now the area of each finite region of the plane partitioned by L'_i is d^2 times the original area. □

*This can be seen in a number of different ways, including taking the bounding box and removing triangles, triangulating the polygon directly, using shoelace formula, etc.

Problem 2.

(proposed by Ethan Tan)

Evaluate

$$\frac{1/2}{1+\sqrt{2}} + \frac{1/4}{1+\sqrt[4]{2}} + \frac{1/8}{1+\sqrt[8]{2}} + \frac{1/16}{1+\sqrt[16]{2}} + \cdots$$

Notes on Marking. This problem essentially had two parts, the first of which could be solved by either rationalising or telescoping. Candidates received 4 points for rationalising and simplifying the numerator, and 5 points for showing that the sum is telescoping. Some candidates scored 2 points for rationalising but leaving the numerator as a product of polynomials. Some candidates lost a point for the wrong answer but the right method.

Solution 1.

(solution by Simeon Kiflie)

Let a_n be the nth term of this series, S_n the nth partial sum, and S the limit of partial sums. By rationalising the denominator of a_n, it can be shown that

$$a_n = \frac{2^{-n}}{2^{2^{-n}}+1} = -2^{-n} \sum_{k=0}^{2^n-1} \left(-2^{2^{-n}}\right)^k.$$

Summing the first n terms with 1 and using the geometric series formula gives

$$S_n + 1 = 2^{-n} \sum_{k=0}^{2^n-1} \left(2^{2^{-n}}\right)^k = \frac{2^{-n}}{2^{2^{-n}}-1}.$$

By substituting $x = 2^{-n}$ and using L'Hôpital's rule, it follows that

$$S + 1 = \lim_{x\to 0} \left(\frac{x}{2^x - 1}\right) = \lim_{x\to 0} \left(\frac{1}{2^x \ln 2}\right) = \frac{1}{\ln 2}$$

and so $S = \dfrac{1}{\ln 2} - 1$. □

Solution 2.

(solution by Julian Yu)

As in solution 1, we find that

$$S_n + 1 = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \left(2^{2^{-n}}\right)^k.$$

However, note that this sum is the same as the lower Riemann sum of the function $f(x) = 2^x$ over the interval $[0, 1]$, with respect to the partition

$$P = [0,\ 1\times 2^{-n},\ 2\times 2^{-n},\ 3\times 2^{-n},\ \ldots,\ 1].$$

Hence, taking the limit as $n \to \infty$, we find that

$$S + 1 = \int_0^1 2^x \, dx$$

and so $S = \dfrac{1}{\ln 2} - 1$. □

Solution 3. *(solution by Ming Yean Lim)*

Observe that

$$\frac{1/2^{n+1}}{1+2^{1/2^{n+1}}} = \frac{1/2^n}{1-2^{1/2^n}} - \frac{1/2^{n+1}}{1-2^{1/2^{n+1}}}.$$

Therefore, the sum is

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1/2^{n+1}}{1+2^{1/2^{n+1}}} &= \lim_{N\to\infty} \sum_{n=0}^{N} \left(\frac{1/2^n}{1-2^{1/2^n}} - \frac{1/2^{n+1}}{1-2^{1/2^{n+1}}} \right) \\
&= \lim_{N\to\infty} -1 - \frac{1/2^{N+1}}{1-2^{1/2^{N+1}}} \\
&= -1 + \lim_{N\to\infty} \frac{1/2^{N+1}}{2^{1/2^{N+1}}-1}.
\end{aligned}$$

As in Solution 1, we may use L'Hôpital's rule to evaluate the last limit, yielding $\frac{1}{\ln(2)} - 1$. $\square$

Problem 3. *(proposed by Ethan Tan)*

A set of points has *point symmetry* if a reflection in some point maps the set to itself. Let $\mathcal{P}$ be a solid convex polyhedron whose orthogonal projections onto any plane have point symmetry. Prove that $\mathcal{P}$ has point symmetry.

Notes on Marking. Many students assumed incorrect properties about projections and the objects they commute with. For instance, it is not true that the vertices of the projection of the polyhedron match one-to-one with the vertices of the polyhedron, because vertices of the polyhedron can project to the interior of the polygon, and become "hidden". In particular, one cannot assume that the projections of vertices as a set of points have point symmetry. It is also not true that the centre of mass commutes with projection, either for the centre of mass of vertices (not all vertices map to vertices of the polyhedron), or for the centre of mass of the entire solid body (this is false for most polyhedra). Some students found characterisations of the centre of the polyhedron conducive to some complete solutions (e.g. midpoint of diameter, or intersection of pre-images of centres of projections), and were awarded marks for these.

Solution 1. *(solution by Ethan Tan)*

Let π_v denote the projection onto a plane perpendicular to v. Then $\pi_v(P)$ has a centre O_v for all v; let ℓ_v, be the line parallel to v passing through O_v.

Let z_{min}, z_{max} be the highest z-coordinates that $\mathcal{P}$ attains. For all v parallel to the xy-plane, ℓ_v has z-coordinate $(z_{\text{min}} + z_{\text{max}})/2$, so they all intersect. Since the choice of z-direction is arbitrary, we see any two ℓ_v intersect. Then ℓ_x, ℓ_y, ℓ_z (parallel to the x, y, z-axes) pairwise intersect, and hence all intersect in the same point O. All other ℓ_v intersect these three lines, and so must also pass through O.

Choose O to be our origin. Suppose for contradiction there is a vertex P of $\mathcal{P}$ such that $-P \notin \mathcal{P}$. Let Π be the plane through P perpendicular to OP. Then $-\Pi \cap \mathcal{P} = X$ is convex (or empty), and does not contain $-P$. Hence there exists a line γ in $-\Pi$ through $-P$, not intersecting $\mathcal{P}$.

Since ℓ_γ passes through O, we have $\pi_\gamma(O)$ is the centre of $\pi_\gamma(\mathcal{P})$. Take this as the origin for $\pi_\gamma(\mathcal{P})$. Then $-\pi_\gamma(P) = \pi_\gamma(-P) \notin \pi_\gamma(\mathcal{P})$, i.e. $\pi_\gamma(\mathcal{P})$ does not have centre $\pi_\gamma(O)$, contradiction. □

Solution 2. *(solution by Tony Wang)*

If a set S is to have point symmetry, then the center of symmetry must be the midpoint of a diameter of S, since otherwise reflection of the diameter will create a greater diameter, which is a contradiction. So, let PQ be a diameter of $\mathcal{P}$ and let M be the midpoint of PQ. We proceed by contradiction.

Let X be a point on the boundary of $\mathcal{P}$ and Y be the second intersection of XM with the boundary of $\mathcal{P}$. Then since we are arguing by contradiction, there must exist an X so that $XM > MY$. Consider the plane perpendicular to XY, and let ℓ be its intersection with the plane of a face that Y is on. This intersection must exist, lest $\mathcal{P}$ not be convex. Project $\mathcal{P}$ orthogonally by ℓ (or any line in ℓ, if it is a plane), and denote this projection by $\cdot'$. Note that $P'Q'$ is still a diameter of $\mathcal{P}'$, and so the center of symmetry of $\mathcal{P}'$ would still have to be M'. However, $\frac{XM}{MY} = \frac{X'M'}{M'Y'}$, and Y' is still on the boundary of $\mathcal{P}'$, whereas X' may now be in the interior of $\mathcal{P}'$. In any case, denoting the second intersection of $Y'M'$ with the boundary of $\mathcal{P}'$

by Z', we actually have $\frac{Z'M'}{M'Y'} \geq \frac{X'M'}{M'Y'} = \frac{XM}{MY} > 1$, and hence $Z'M' > M'Y'$. This implies that $\mathcal{P}'$ cannot be point symmetric, a contradiction. $\square$

Solution 3. *(solution by Harun Khan)*

Consider an edge UV of the polyhedron where U, V are vertices. Consider a plane P_{UV} not parallel to any face of $\mathcal{P}$ and containing UV such that $P_{UV} \cap \mathcal{P} = UV$. Project along a line parallel to P_{UV} and perpendicular to UV, and call this projection π_1. As all projections are point symmetric, there exist two unique vertices $U', V' \in \mathcal{P}$ such that $\pi_1(U')$ and $\pi_1(V')$ is point symmetric to $\pi_1(U)$ and $\pi_1(V)$, respectively. We can draw a plane P' parallel to P_{UV} and containing $U'V'$ where $P' \cap \mathcal{P} = U'V'$ because P_{UV} is not parallel to any face of $\mathcal{P}$.

We will now show that $U'V'$ is parallel and equal in length to UV. We project along a different line contained in P_{UV} and not parallel to UV or $U'V'$. This projection by π_2 takes U, V to $\pi_2(U), \pi_2(V)$ which is symmetric to some $\pi_2(U)', \pi_2(V)'$ respectively in the plane. Inverting π_2 at $\pi_2(U)', \pi_2(V)'$ we get a point U'', V'' that must lie in $P' \cup \mathcal{P} = U'V'$. (Here we use the fact that two parallel edges in a point-symmetric polygon have symmetric endpoints.) Hence, since orientation is preserved, $U'' = U'$ and $V'' = V'$. This argument shows only one such possible line segment works, (and a line segment equal in length and parallel to UV would work) hence $U'V'$ must be parallel and equal in length to UV.

The last caveat is that vertices belonging to two edges might be symmetric to different points for each edge. Let's show this is not possible. Take two adjacent edges UV and VW. Take the projection π along the line of intersection between the two planes P_{UV} and P_{VW} as our choice of π_2 for both UV and VW. Hence this projection 'glues' $U'V'$ to $V'W'$. This proves the existence of U', V', W' such that the midpoint of UU', the midpoint of VV' and the midpoint of WW' are all the same. Applying this argument transitively to all vertices of $\mathcal{P}$ finishes the problem. $\square$

Problem 4. *(proposed by Tony Wang and Ethan Tan)*

Fix a set of integers S. An integer is *clean* if it is the sum of distinct elements of S in exactly one way, and *dirty* otherwise. Prove that the set of dirty numbers is either empty or infinite.

Note: We consider the empty sum to equal 0.

Notes on Marking. Candidates who attempted this problem received 1 point for proving that if an integer has two representations then there are infinitely many dirty integers (which is needed in solutions 2 and 3). Some candidates were awarded an additional point for showing that if a, b are clean with $|a-b| = s$ for $s \in S$ then a and b differ in s-dependence, or statements similar in nature. No marks were awarded for proving that S must be infinite.

Solution 1. *(solution by Tony Wang)*

We proceed by contradiction: suppose that the set of dirty numbers is non-empty but finite, and let the greatest dirty number be M. For each clean number n, let the set of distinct elements of S that sum to n be called its *representation*. Let n be called *a-dependent* if a is one of those distinct elements, and *a-free* otherwise. Now, if n and $n + a$ are both clean, then they must differ in their a-dependence, for if both are a-dependent then we can remove a from $n + a$ to achieve a second representation of n, and if both are a-free we can add a to the representation of n to achieve a second representation of $n + a$.

We now prove an important lemma: for distinct $a, b \in S$, $\nu_2(a) \neq \nu_2(b)$. To prove this, suppose that $a \neq b$ but $\nu_2(a) = \nu_2(b) = e$. Let N be an a, b-free number much greater than M (if N contains a or b in its representation, then we can simply remove them to obtain a new N). Now, letting $d = \text{lcm}(a, b)$, note that if $n > M$, then n and $n + 3d$ must differ in both a- and b-dependence, since $3d$ is an odd multiple of both a and b. Then, if $n+3d$ is a, b-dependent, then $n + 3d - a - b$ is a, b-free. So adding $3d - a - b > 0$ to an a, b-free number gives a new a, b-free number. This means that $N + \frac{d}{2^e}(3d - a - b)$ is a, b-free, but it is also equal to $N + \frac{3d-a-b}{2^e}d$, which is a, b-dependent, a contradiction.

To finish, let W be a dirty number with the greatest 2-adic valuation, and let $w = \nu_2(W)$. It follows that every integer with a greater 2-adic valuation is clean. We know from the above lemma that there is at most one element $x \in S$ with with $\nu_2(x) = w$, but since some integers of 2-adic valuation w are clean, x must exist. Now consider the representations of W: any such representation must contain x as the member of the representation with the smallest 2-adic valuation. Removing x from these representations, we obtain representations of $W - x$. But we know $W - x$ to be clean, since $\nu_2(W - x) > w$, and so W has exactly one representation: the union of x with the representation of $W - x$. Thus W is clean, a contradiction. □

Solution 2. *(solution by contestants)*

We start in the same way as solution 1: importantly, we exploit the fact that if n and $n + a$ are both clean, then they must differ in their a-dependence.

Now pick a dirty $x \in \mathbb{Z}$. Since S is infinite and the set of dirty numbers is finite, there must exist $s \in S$ such that both $x - s$ and $x + s$ are clean. Likewise, there must exist some $t \in S$ such that $x - s + t, x + t$, and $x + s + t$ are all clean and t is not in the representation of either $x - s$ or $x + s$. Note that no integer can have more than one representation. If some $y \in \mathbb{Z}$ has multiple representations

$$y = a_1 + \ldots + a_n = b_1 + \ldots + b_m,$$

then for any $u \in S \backslash \{a_1, \ldots, a_n, b_1, \ldots, b_m\}$, $y + u$ will have multiple representations. Hence there are an infinite number of dirty integers. Now assume that no integer has multiple representations.

Hence $x - s$ must be s-dependent, as otherwise we would get a representation of x. Similarly, $x + s$ must be s-free. Now $x - s + t$ must be s- and t-dependent, and $x + s + t$ must be t-dependent but s-free. But now $x + t$ must be s-dependent (going down from $x + s + t$), but also s-free (going up from $x - s + t$), a contradiction.

Thus, there are either no dirty numbers or an infinite number of them. □

Solution 3.

(solution by contestants)

As in solution 2, if some $y \in \mathbb{Z}$ has multiple representations

$$y = a_1 + \ldots + a_n = b_1 + \ldots + b_m,$$

then for any $u \in S \backslash \{a_1, \ldots, a_n, b_1, \ldots, b_m\}$, $y + u$ will have multiple representations. Hence there are an infinite number of dirty integers.

Thus we can assume that no integer has multiple representations. Clearly S is infinite and countable, so write $S = \{s_1, s_2, \ldots\}$. Denote by S_n to be the set of sum of elements of subsets of $\{s_1, \ldots, s_n\}$. Note that for any $n < m$, we have $S_n \subset S_m$. Suppose by contradiction that the set of dirty numbers is nonempty but finite. Let a and b be the smallest and largest dirty numbers, respectively. For a finite $A \subset \mathbb{Z}$ denote by $\text{range}(A) = \max(A) - \min(A)$. Clearly the sequence $\text{range}(S_n)$ tends to infinity as n goes to infinity.

Let $c_1 = 2(b - a)$. Let $N_2 \in \mathbb{N}^*$ such that $\text{range}(S_{N_2}) > 2c_1$ and let $c_2 = \text{range}(S_{N_2}) + 1$. We will prove by induction the following:

- **Lemma:** For all $n \geq N_2, \forall k \in S_n, \exists k' \in S_n$ such that $c_1 < |k' - k| < c_2$.

  Proof. Let us start with $n = N_2$ and fix $k \in S_n$. We have that

  $$\text{range}(S_n) = \max(S_n) - \min(S_n) = (\max(S_n) - k) + (k - \min(S_n)).$$

  Clearly at least one of $\max(S_n) - k$ and $k - \min(S_n)$ are at least $\text{range}(S_n)/2$, thus there exists some $k' \in S_n$ such that $|k' - k| \geq \text{range}(S_n)/2 > c_1$. Furthermore, we obviously have $|k' - k| < c_2$ (as this happens for any difference of elements of S_{N_2}).

  Now suppose that we proved the above for some $n \geq N_2$ and we will prove it for $n+1$. Fix $k \in S_{n+1}$. k is clearly clean. If k contains s_{n+1} in its representation, then $k - s_{n+1} \in S_n$, otherwise $k \in S_n$. If $k - s_{n+1} \in S_n$, then by induction hypothesis there is some $k' \in S_n$ such that $c_1 < |k' - (k - s_{n+1})| < c_2$. Then $k' + s_{n+1} \in S_{n+1}$ and $c_1 < |(k' + s_{n+1}) - k| < c_2$. This finishes the induction. □

Now take $M = 100c_2$ and let $N_1 \in \mathbb{N}^*$ such that

$$a - 1, \ldots, a - M, b + 1, \ldots, b + M \in S_{N_1}$$

(this exists as all of those numbers must be clean). Let c be a dirty integer, and pick $N \in \mathbb{N}^*$ such that $N > N_1$, $N > N_2$ and $s_N > b - c + M$ (N exists as S is clearly unbounded). Then we have that

$$a - 1, \ldots, a - M, b + 1, \ldots, b + M \in S_{N_1} \subseteq S_{N-1} \implies$$
$$a + s_N - 1, \ldots, a + s_N - M, b + s_N + 1, \ldots, b + s_N + M \in S_N.$$

Furthermore, we have $c + s_N > c + b - c + M = b + M > b$, hence $c + s_N$ is clean. Since c is dirty, $c + s_N \notin S_N$. Let K be such that $c + s_N \notin S_{K-1}$ and $c + s_N \in S_K$. Clearly we have $K > N$. Hence $c + s_N - s_K \in S_{K-1}$.

Now apply to this the previous lemma, and we will get some $k \in S_{K-1}$ such that $c_1 < |(c + s_N - s_K) - k| < c_2$. Since $2(b-a) < |(s_K+k)-(c+s_N)|$, then either $(s_K+k)-(c+s_N) > 2(b-a)$, or $(s_K+k)-(c+s_N) < -2(b-a)$. In the first case, we get $s_K+k > 2(b-a)+c+s_N \geq b+s_N$. If $s_K + k > b + s_N + M$, then $(s_K + k) - (c + s_N) > b + s_N + M - c - s_N \geq M = 100c_2 > c_2$, contradiction. Similarly, in the second case, we get $s_K + k < -2(b - a) + c + s_N \leq a + s_N$. If $s_K + k < a + s_N - M$, then $(s_K + k) - (c + s_N) < a + s_N - M - c - s_N \leq -M = -100c_2 < -c_2$, contradiction.

Thus $s_K + k$ is between $a + s_N - 1$ and $a + s_N - M$ or between $b + s_N + 1$ and $b + s_N + M$. Hence $s_K + k$ is either some $a + s_N - t$ or some $b + s_N + t$ for some $1 \leq t \leq M$. But $k \in S_{K-1}$, hence $s_K + k$ has a representation containing s_K, but as $a + s_N - t$ and $b + s_N + t$ are all in S_N, it also has a representation without s_K (as $K > N$). This is a contradiction, which finishes the proof. □

Problem 5. *(proposed by Ethan Tan)*

A robot on the number line starts at 1. During the first minute, the robot writes down the number 1. Each minute thereafter, it moves by one, either left or right, with equal probability. It then multiplies the last number it wrote by n/t, where n is the number it just moved to, and t is the number of minutes elapsed. It then writes this number down. For example, if the robot moves right during the second minute, it would write down $2/2 = 1$.

Find the expected sum of all numbers it writes down, given that it is finite.

Notes on Marking. A mark was awarded for considering linearity of expectation, observing the similarity to a Taylor series expansion, or considering generating functions.

Solution 1. *(solution by Ethan Tan)*

Instead of letting the robot go left and right with equal probability, we allow it to split into two robots, one going left and one going right. Each robot, upon reaching n, then clones itself to become n robots. It is clear by linearity of expectation that the answer to the original problem is now

$$\sum_{t=1}^{\infty} \frac{r_t}{2^{t-1} \cdot t!},$$

where r_t is the number of robots at the end of the t^{th} second. ($r_1 = 1$, and the 2^{t-1} term comes from replacing moving left or right with equal chance by splitting into two.)

We also let the robot have two arms, each of length 1, on either side of the robot. A robot at n at time t becomes $n-1$ robots at $n-1$ and $n+1$ robots at $n+1$ at time $t+1$. This robot at time t contributes one hand at $n-1$ and one hand at $n+1$, which at time $t+1$ becomes $n-1$ hands at $n-2$, $(n-1)+(n+1)$ hands at n, and $n+1$ hands at $n+2$.

So we can see that any hand at position k at time t becomes in k hands at $k-1$ and k hands at $k+1$ at time $t+1$. We now encode this information with the following key identity:

$$\boxed{\frac{d}{dx}u^n = n(u^{n-1} + u^{n+1}), \text{ where } u = \tan x.} \tag{†}$$

At time t, assign each hand at position k with the value $u^k = \tan^k(x)$, and let the total value at time t be $v_t(x)$. Note $\tan(\pi/4) = 1$, so

$$v_t(\pi/4) = (\#\text{ hands at time } t) = 2r_t.$$

But we know by (†) that $v_{t+1}(x) = v_t'(x)$, so that $v_{t+1}(x) = \frac{d^t}{dx^t}v_1(x)$. Let $f(x) = \tan(x)$; we have $v_1(x) = 1 + \tan^2(x) = f'(x)$. Then

$$\sum_{t=1}^{\infty} \frac{r_t}{2^{t-1} \cdot t!} = \frac{1}{2}\sum_{t=1}^{\infty} \frac{v_t(\pi/4)}{t!}\left(\frac{1}{2}\right)^{t-1} = \sum_{t=1}^{\infty} \frac{f^{(t)}(\pi/4)}{t!}\left(\frac{1}{2}\right)^{t} = \left[\sum_{t=0}^{\infty} \frac{f^{(t)}(\pi/4)}{t!}\left(\frac{1}{2}\right)^{t}\right] - f(\pi/4),$$

which by Taylor expansion is equal to

$$\begin{aligned} f\left(\frac{\pi}{4} + \frac{1}{2}\right) - f(\pi/4) &= \tan\left(\frac{\pi}{4} + \frac{1}{2}\right) - 1 \\ &= \sec(1) + \tan(1) - 1. \end{aligned}$$

□